Using Leslie matrices as application of eigenvalues and eigenvectors in a first course in Linear Algebra

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Chapter 5 Eigenvalues and Eigenvectors

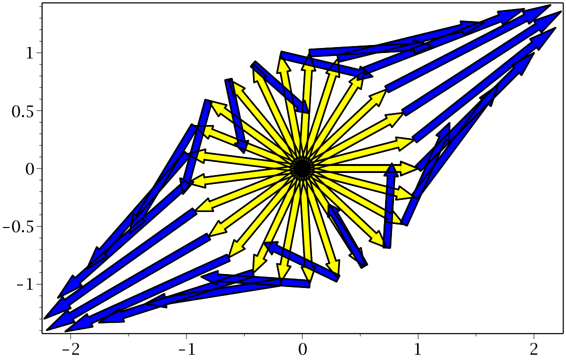
D. Lay, S. Lay and J. McDonald. *Linear Algebra and its Applications*, 5th ed, Pearson, 2016.

- 5.1 Eigenvalues and Eigenvectors

Definition. A non-zero vector v is an **eigenvector** of an n by n matrix A if $Av = \lambda v$ for some scalar λ called an **eigenvalue** of A corresponding to v.

Example. Is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$? If yes what is λ?

Now what? Properties then Method?

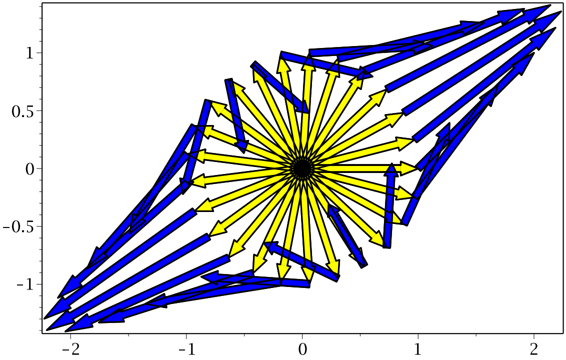


$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence $Av \neq \lambda v$.

Can you see the eigenvalues and eigenvectors of A?



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Can you see the eigenvalues and eigenvectors of A?
I see $\lambda_1 = 1.5$, $v_1 = [1.5, 1]$ and $\lambda_2 = -0.6$, $v_2 = [0.5, -1]$.

Chapter 5 Eigenvalues and Eigenvectors

- 5.1 Eigenvalues and Eigenvectors
- 5.2 The Characteristic Polynomial
- 5.3 Diagonalization
- 5.4 Complex Eigenvalues

What application should we use choose?

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What application should we use choose?

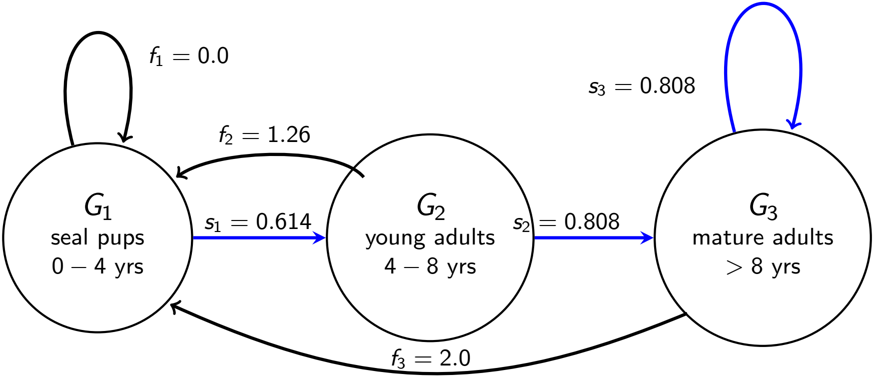
- It must fit in one 45–50 minute lecture
- It must be easy to understand – no physics
- It must not require additional mathematics – no ODEs

Outline

- The Leslie age distribution model
- Exercises
- More on Leslie matrices
- Exercises
- Population control and harvesting
- Exercises

The Leslie Age Distribution Model

Divide the female population into n age groups $G_1, G_2, \ldots, G_n$.
Let s_i be the survival rate of age group i.
Let f_i be the average no. of females born to an individual in G_i.



The Leslie Age Distribution Model

Let p_i^t be the population of G_i at time t.

Let $P^{(t)} = [p_1^t, p_2^t, \ldots, p_n^t]$ be the population vector at time t.

$$P^{(t+1)} = \left[\begin{array}{c} f_1 p_1^t + f_2 p_2^t + f_3 p_3^t \\ s_1 p_1^t \\ s_2 p_2^t + s_3 p_3^t \end{array} \right]$$

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Thus $P^{(t+1)} = LP^{(t)}$. L is called a Leslie matrix.

Patrick H. Leslie. The use of matrices in certain population mathematics. *Biometrika* **33**(3): 183–212, 1945.

Sable Island Grey Seals and Northern Spotted Owls

Grey Seals (4 years)

$f_1 = 0.0 \quad f_2 = 1.26 \quad f_3 = 2.0$

$s_1 = .614 \quad s_2 = .808 \quad s_3 = .808$

$$\begin{bmatrix} 0.0 & 1.26 & 2.0 \\ .614 & 0 & 0 \\ 0 & .808 & .808 \end{bmatrix}$$

Spotted Owls (1 year)

$f_1 = 0.0 \quad f_2 = 0.0 \quad f_3 = 0.33$

$s_1 = 0.18 \quad s_2 = 0.71 \quad s_3 = 0.94$

$$\begin{bmatrix} 0.0 & 0.0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{bmatrix}$$

What happens to the seal and owl populations?

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What happens to the seal and owl populations?

Answer: $\lambda^+ = 1.49$ for the seals and $\lambda^+ = 0.91$

What happens to the Grey Seal population?

Suppose $P^{(0)} = [1, 1, 1]$ (thousands). According to the model

$$P^{(1)} = LP^{(0)} = \begin{bmatrix} 0.0 & 1.26 & 2.0 \\ .614 & 0 & 0 \\ 0 & .808 & .808 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.26 \\ .614 \\ 1.616 \end{bmatrix}$$

$$P^{(2)} = LP^{(1)} = \begin{bmatrix} 0.0 & 1.26 & 2.0 \\ .614 & 0 & 0 \\ 0 & .808 & .808 \end{bmatrix} \begin{bmatrix} 3.26 \\ .614 \\ 1.616 \end{bmatrix} = \begin{bmatrix} 4.00564 \\ 2.00164 \\ 1.80184 \end{bmatrix}$$

The population is growing!
After 8 years it's $4.00+2.00+1.80 = 7.8$ thousand.

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What is happening to the age distribution?

The Age Distribution

Let $pop(t) = p_1^t + p_2^t + \cdots + p_n^t$ be the total pop. at time t.
Let $D^{(t)} = P^{(t)}/pop(t)$ be the age distribution at time t.

t	$P^{(t)}$	$D^{(t)}$
0	[1.00, 1.00, 1.00]	[0.333, 0.333, 0.333]
1	[3.26, .614, 1.616]	[0.5938069217, 0.1118397086, 0.2943533698]
2	[4.005, 2.001, 1.801]	[0.5129438400, 0.2563208146, 0.2307353453]
15	[747.5, 307.7, 363.6]	[0.5268357541, 0.2168591050, 0.2563051409]
16	[1115., 459.0, 542.5]	[0.5268357429, 0.2168591067, 0.2563051504]

The population age distribution has converged to

$G_1 = 0.5268357$ (seal pups 0–4 yrs)
$G_2 = 0.2168591$ (young adults 4–8 yrs)
$G_3 = 0.2563051$ (mature adults ≥ 8 yrs)

$$
\begin{array}{ccc}
P^{(16)} & P^{(15)} & P_i^{(16)}/P_i^{(15)} \\
\begin{bmatrix} 1115.126895 \\ 459.0148362 \\ 542.5083059 \end{bmatrix} &
\begin{bmatrix} 747.5811664 \\ 307.7235768 \\ 363.6975940 \end{bmatrix} &
\begin{array}{c} 1.491646586 \\ 1.491646630 \\ 1.491646673 \end{array}
\end{array}
$$

We have $P^{(16)} = 1.491646\, P^{(15)}$ to 7 decimal places!
But $P^{(16)} = L\, P^{(15)} = 1.491646\, P^{(15)}$. Therefore $P^{(15)}$ is an eigenvector of L with eigenvalue $\lambda^{+} = 1.491646$. Since $D^{(15)} = P^{(15)}/pop(P^{(15)})$, $D^{(15)}$ is an eigenvector of L.

$\lambda^{+} = 1.49646$ tells us the growth rate of the population.
The eigenvector $D^{(15)}$ tells us the long term age distribution.

What happens for Leslie Matrices

Theorem

For any non-zero initial population $P^0 = [p_1^0, p_1^0, \ldots, p_n^0]$, if at least one fertility rate f_i is positive, the Leslie matrix L has a unique positive eigenvalue λ^+. If v^+ is a corresponding eigenvector and at least two consecutive fertility rates are positive, λ^+ is dominant and the population distribution will converge to an eigenvector of L, that is $\lim_{t\to\infty} D^{(t)}$ exists and is a multiple of v^+.

We also have the following physical interpretation for λ^+.

$$\begin{array}{ll} \lambda^+ < 1 & \text{means the population will decline exponentially.} \\ \lambda^+ > 1 & \text{means the population will grow exponentially.} \\ \lambda^+ = 1 & \text{means the population is stable, it does not change.} \end{array}$$

Exercises

3 For the Leslie matrix below calculate the eigenvalues. You should find that one is 0 and one is positive. For the positive eigenvalue, determine the corresponding eigenvector. What is the long term population distribution vector?

$$L = \begin{bmatrix} 0 & 7/6 & 7/6 \\ 1/2 & 0 & 0 \\ 0 & 2/3 & 2/3 \end{bmatrix}$$

4 For the northern spotted owl population (see Figure 2), starting with $P^0 = [0.2, 0.1, 0.7]$, calculate $P^4 = L^4 P^0$ and $P^5 = L^5 P^0$ and determine the age distribution. To how may decimal places has the population distribution converged. Estimate the corresponding eigenvalue.

Exercises (cont.)

5 This exercise is taken from Poole [8]. Woodland caribou are found primarily in western Canada and the American northwest. The fertility rates and survival rates are given in the table below. The data shows that caribou cows do not give birth during their first two years and the survival rate for caribou calves is low.

Age	0–2	2–4	4–6	6–8	8–10	10–12	12–14
f_i	0.0	0.2	0.9	0.9	0.9	0.8	0.3
s_i	0.3	0.7	0.9	0.9	0.9	0.6	0.0
$P_i^{(0)}$	10	2	8	5	12	0	1

Construct the Leslie matrix. Shown also in the last row is the female caribou population in Jasper National park in 1990. Predict the female population in 1992, 1994, 1996, 1998 and 2000. What do you conclude will happen to the population in the long term? Use a computer to compute the eigenvalues of L. What is λ^+? What does this tell you about the population?

More on Leslie Matrices

A Leslie matrix is an n by n matrix of the form

$$L = \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix} \tag{1}$$

where $n \geq 2$, the survival rates $s_i > 0$ and fertility rates $f_i \geq 0$ with at least one $f_i > 0$. Notice $s_n = 0$.

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Some Facts about Leslie Matrices

The characteristic polynomial $C(\lambda) = \det(\lambda I - L)$ is given by

$$\lambda^n - f_1\lambda^{n-1} - s_1 f_2 \lambda^{n-2} - s_2 s_1 f_3 \lambda^{n-3} - \cdots - s_{n-1} \ldots s_3 s_2 s_1 f_n$$

To show that L has one positive eigenvalue λ^+ consider

$$q(\lambda) = \frac{f_1}{\lambda} + \frac{f_2 s_1}{\lambda^2} + \frac{f_3 s_1 s_2}{\lambda^3} + \cdots + \frac{f_n s_1 s_2 \cdots s_{n-1}}{\lambda^n}$$

Since $q(\lambda)$ is monotonically decreasing and $q(\lambda) = 1$ (exercise) it follows that there is only one positive λ^+. The eigenvector is

$$v^+ \; = \; \left[\begin{array}{cccccc} 1 & \frac{b_1}{\lambda} & \frac{b_1 b_2}{\lambda^2} & \frac{b_1 b_2 b_3}{\lambda^3} & \ldots & \frac{b_1 b_2 b_{n-1}}{\lambda^{n-1}} \end{array} \right]^T$$

Exercises

2 Show that $q(\lambda) = 1$.

3 Show that the positive eigenvalue λ^+ of a Leslie matrix has algebraic multiplicity 1. Hint: a root λ^+ of a polynomial $q(\lambda)$ has multiplicity 1 if and only if $q'(\lambda^+) \neq 0$.

5 The net reproduction rate of a population is defined as

$$r = f_1 + f_2 s_1 + f_3 s_1 s_2 + \cdots + f_n s_1 s_2 \ldots s_{n-1}.$$

Explain why r can be interpreted as the average number of daughters born to a female over her lifetime. It follows that if $r > 1$ the population will grow but if $r < 1$ it will decline. Calculate r for caribou population in Section 2 Exercise 5.

Population stabilization and harvesting

Consider the Leslie matrix for the grey seal population.

$$\begin{bmatrix} 0 & 1.26 & 2.0 \\ 0.614 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

We have determined that $\lambda^+ = 1.49$ which means the seal population is growing by almost 50% every four years.

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We have determined that $\lambda^+ = 1.49$ which means the seal population is growing by almost 50% every four years.
Idea: To stabilize the population we want to pick $s_1, s_2, s_3, f_1, f_2, f_3$ so that $\lambda^+ = 1$.

1. Reduce s_1 by culling the seal pups every 4 years.
2. Reduce all f_i by shooting all seals with infertility darts.
3. Harvest the adult seals (≥ 4 yrs).

Choose s_1 such that

$$L = \begin{bmatrix} 0 & 1.26 & 2.0 \\ s_1 & 0 & 0 \\ 0 & 0.808 & 0.808 \end{bmatrix}$$

has eigenvalue $\lambda^+ = 1$. Let

$$I - L = \begin{bmatrix} 1 & -1.26 & -2.0 \\ -s_1 & 1 & 0 \\ 0 & -0.808 & .192 \end{bmatrix}$$

$$\det(I - L) = -0.192 + 1.85792\, s_1$$

Solving $\det(I - L) = 0$ I get $s_1 = 0.1033413710$.

Exercises

1. For the grey seal population, what proportion of the seals must we render infertile to stabilize the population.
2. For the grey seal population what is the maximum sustainable harvesting rate assuming we do not harvest seal pups. Why might it be dangerous to harvest at this rate?
3. For the northern spotted owl population what must s_1 be so that the owl population stabilizes? Comment on the stability of the population.
4. If the government tries to eradicate northern owl predators so that all s_1, s_2, s_3 increase, what rate must they increase by to stabilize the population?

Questions?

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