MACM 401/MATH 701/MATH 819/CMPT 881 Assignment 6. Spring 2011.

Michael Monagan

This assignment is due 4:00pm Thursday April 14th.
For problems involving Maple calculations and Maple programming, you should submit a printout of a Maple worksheet of your Maple session.
Late Penalty: -20% for up to 24 hours late. Zero after that.

Question 1 (15 marks): Symbolic Integration

Implement a Maple procedure `INT` that evaluates and simplifies indefinite integrals $\int f(x)\mathrm{d}x$.
For a constant c your Maple procedure should apply the following rules

$$\int c\,\mathrm{d}x \to cx.$$

$$\int cf(x)\,\mathrm{d}x \to c\int f(x)\,\mathrm{d}x.$$

$$\int f(x)+g(x)\,\mathrm{d}x \to \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

$$\int x^{-1}\,\mathrm{d}x = \ln x.$$

$$\text{For } c \neq -1, \quad \int x^c\,\mathrm{d}x = \frac{x^{c+1)}}{c+1}.$$

$$\int e^x\,\mathrm{d}x = e^x.$$

$$\int x^n e^x\,\mathrm{d}x \to x^n\,e^x - \int nx^{n-1}e^x\,\mathrm{d}x.$$

$$\int \ln x\,\mathrm{d}x = x\ln x - x.$$

$$\text{For } n \geq 1, \quad \int x^n \ln x\,\mathrm{d}x \to ??$$

You may ignore the constant of integration. NOTE: e^x in Maple is `exp(x)`, i.e. it's a function not a power. HINT: use the `diff` command for differentiation to determine if a Maple expression c is a constant w.r.t. x. Test your program on the following.

```
> INT( x^2 + 2*x + 3, x );
> INT( x^(-1) + 2*x^(-2) + 3*x^(1/2), x );
> INT( exp(y) + ln(y) + cos(x), y );
> INT( 2*f(x) + 3*y*x/2 + 3*ln(2), x );
> INT( x^3*exp(x) + 2*x^2*ln(x) - sin(2*x), x );
```

Question 2: Non-elementary Integrals (20 marks)

Apply the Risch algorithm to prove that the following integrals are not elementary.
Reference: 12.6, 12.7

$$\int \frac{e^x}{x} \mathrm{d}x$$

$$\int \frac{1}{\log(x)^2} \mathrm{d}x$$

$$\int \frac{\log(x)}{x+1} \mathrm{d}x$$

$$\int e^x \log(x) \mathrm{d}x$$

Question 3: Elementary Integrals (20 marks)

Apply the Risch algorithm to compute the following elementary integrals.
Reference: 12.6, 12.7

$$\int \frac{e^{2x}}{e^{2x} + e^x + 1} \, \mathrm{d}x$$

$$\int 2\theta + 2 - \frac{1/x + 1}{(\theta + x)^2} + \frac{1}{x\theta} \, \mathrm{d}x \qquad \text{where} \quad \theta = \log(x)$$

$$\int \theta + x\theta + \frac{2}{x}\theta^2 - \frac{1}{x^2}\theta^2 \, \mathrm{d}x \qquad \text{where} \quad \theta = e^x$$

Question 4: Cost of the linear p-adic Newton iteration (15 marks)

Let $a \in \mathbb{Z}$ and $u = \sqrt{a}$. Suppose $u \in \mathbb{Z}$. The linear P-adic Newton iteration for computing u from $u \bmod p$ that we gave in class is based on the following linear p-adic update formula:

$$u_k = -\frac{\phi_p(f(u^{(k)})/p^k)}{f'(u_0)} \bmod p.$$

where $f(u) = a - u^2$. A direct coding of this update formula for the $\sqrt{}$ problem in $\mathbb{Z}$ led to the code below where I've modified the algorithm to stop if the error $e < 0$ instead of using a lifting bound B.

```
ZSQRT := proc(a,u0,p) local U,pk,k,e,uk,i;
    u := mods(u0,p);
    i := modp(1/(2*u0),p);
    pk := p;
    for k do
        e := a - u^2;
        if e = 0 then return(u); fi;
        if e < 0 then return(FAIL) fi;
        uk :=  mods( iquo(e,pk)*i, p );
        u := u + uk*pk;
        pk := p*pk;
    od;
end:
```

The running time of the algorithm is dominated by the squaring of `u` in `a := a - u^2` and the long division of `u` by `pk` in `iquo(e,pk)`. Assume the input a is of length n base p digits. At the beginning of iteration k, $u = u^{(k)} = u_0 + u_1 p + ... + u_{k-1}p^{k-1}$ is an integer of length at most k base p digits. Thus squaring `u` costs $\mathrm{O}(k^2)$ (assuming classical integer arithmetic). In the division of `e` by `pk` $= p^k$, `e` will be an integer of length n base p digits. Assuming classical integer long division is used, this division costs $O((n-k+1)k)$. Since the loop will run $k = 1, 2, ..., n/2$ for the $\sqrt{}$ problem the total cost of the algorithm is dominated by

$$\sum_{k=1}^{n/2} \left(k^2 + (n-k+1)k\right) \ \in \ O(n^3).$$

Redesign the algorithm so that the overall time complexity is $O(n^2)$ assuming classical algorithms for integer arithmetic. Now implement your algorithm in Maple and verify that it works correctly and that the running time is $\mathrm{O}(n^2)$. Use the prime $p = 9973$.

Hint 1: $e = a - u^2 = a - u^{(k)^2} = a - (u^{(k-1)} + u_{k-1}p^{k-1})^2 = (a - u^{(k-1)^2}) - 2u^{k-1}u_{k-1}p^{k-1} - u_{k-1}^2 p^{2k-2}$. Notice that $a - u^{(k-1)^2}$ is the error that was computed in the previous iteration.
Hint 2: We showed that the algorithm for computing the p-adic representation of an integer is $\mathrm{O}(n^2)$. Notice that it does not divide by p^k, rather, it divides by p each time round the loop.