MACM 401/MATH 701/MATH 801
Assignment 4, Spring 2019.

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Due Friday March 8th by 4pm. Hand in to dropoff box 1a outide AQ 4100.
For problems involving Maple calculations and Maple programming, you should submit a printout of a Maple worksheet of your Maple session.
Late Penalty: -20% for up to 72 hours late. Zero after that.

Note, you may use Maple for all calculations unless asked to do the question by hand.

Question 1: P-adic Lifting (25 marks)

Reference: Section 6.2 and 6.3.

(a) By hand, determine the p-adic representation of the integer $u = 116$ for $p = 5$, first using the positive representation, then using the symmetric representation for $\mathbb{Z}_5$.

(b) Theorem 2: Let $u, p \in \mathbb{Z}$ with $p > 2$. For simplicity assume p is odd.
If $-\frac{p^n}{2} < u < \frac{p^n}{2}$ there exist unique integers $u_0, u_1, \ldots, u_{n-1}$ such that $u = u_0 + u_1 p + \cdots + u_{n-1} p^{n-1}$ and $-\frac{p}{2} < u_i < \frac{p}{2}$.

Prove uniqueness.

(c) Determine the cube-root, *if it exists*, of the following polynomials

$$a(x) = x^6 - 531x^5 + 94137x^4 - 5598333x^3 + 4706850x^2 - 1327500x + 125000,$$

$$b(x) = x^6 - 406\,x^5 + 94262\,x^4 - 5598208\,x^3 + 4706975\,x^2 - 1327375\,x + 125125$$

using reduction mod 5 and linear p-adic lifting. You will need to derivive the update formula by modifying the update formula for computing the $\sqrt{a(x)}$.

Factor the polynomials so you know what the answers are. Express your the answer in the p-adic representation. To calculate the initial solution $u_0 = \sqrt[3]{a} \bmod 5$ use any method. Use Maple to do all the calculations.

Question 2: Hensel lifting (15 marks)

Reference: Section 6.4 and 6.5.

(a) Given

$$a(x) = x^4 - 2\,x^3 - 233\,x^2 - 214\,x + 85$$

and image polynomials

$$u_0(x) = x^2 - 3\,x - 2 \quad \text{and} \quad w_0(x) = x^2 + x + 3,$$

satisfying $a \equiv u_0\, w_0 \pmod 7$, lift the image polynomials using Hensel lifting to find (if there exist) u and w in $\mathbb{Z}[x]$ such that $a = uw$.

(b) Given

$$b(x) \;=\; 48\,x^4 - 22\,x^3 + 47\,x^2 + 144$$

and an image polynomials

$$u_0(x) = x^2 + 4\,x + 2 \quad \text{and} \quad w_0 = x^2 + 4\,x + 5$$

satisfying $b \;\equiv\; 6\,u_0\,w_0 \pmod 7$, lift the image polynomials using Hensel lifting to find (if there exist) u and w in $\mathbb{Z}[x]$ such that $b \;=\; uw$.

Question 3: Determinants (25 marks)

Consider the following 3 by 3 matrix A of polynomials in $\mathbb{Z}[x]$ and its determinant d.

```
> P := () -> randpoly(x,degree=2,dense):
> A := Matrix(3,3,P);
```

$$A := \begin{bmatrix} -55 - 7\,x^2 + 22\,x & -56 - 94\,x^2 + 87\,x & 97 - 62\,x \\ -83 - 73\,x^2 - 4\,x & -82 - 10\,x^2 + 62\,x & 71 + 80\,x^2 - 44\,x \\ -10 - 17\,x^2 - 75\,x & 42 - 7\,x^2 - 40\,x & 75 - 50\,x^2 + 23\,x \end{bmatrix}$$

```
> d := LinearAlgebra[Determinant](A);
```

$$d := -224262 - 455486\,x^2 + 55203\,x - 539985\,x^4 + 937816\,x^3 + 463520\,x^6 - 75964\,x^5$$

(a) (15 marks) Let A by an n by n matrix of polynomials in $\mathbb{Z}[x]$ and let $d = \det(A)$. Develop a modular algorithm for computing $d = \det(A) \in \mathbb{Z}[x]$. Your algorithm will compute determinants of A modulo a sequence of primes and apply the CRT. For each prime p it will compute the determinant in $\mathbb{Z}_p[x]$ by evaluation and interpolation. In this way we reduce computation of a determinant of a matrix over $\mathbb{Z}[x]$ to many computations of determinants of matrices over $\mathbb{Z}_p$, a field, for which ordinary Gaussian elimination, which does $O(n^3)$ arithmetic operations in $\mathbb{Z}_p$, may be used.

You will need bounds for $\deg d$ and $||d||_\infty$. Use primes $p = [101, 103, 107, ...]$ and use Maple to do Chinese remaindering. Use $x = 1, 2, 3, ...$ for the evaluation points and use Maple for interpolation. Implement your algorithm in Maple and test it on the above example.

To reduce the coefficients of the polynomials in A modulo $p = 7$ in Maple use

```
> B := A mod p;
```

To evaluate the polynomials in B at $x = \alpha$ modulo p in Maple use

```
> C := eval(B,x=alpha) mod p;
```

To compute the determinant of a matrix C over $\mathbb{Z}_p$ in Maple use

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> Det(C) mod p;
```

(b) (10 marks) Suppose A is an n by n matrix over $\mathbb{Z}[x]$ and $A_{i,j} = \sum_{k=0}^{d} a_{i,j,k} x^k$ and $|a_{i,j,k}| < B^m$. That is A is an n by n matrix of polynomials of degree at most d with coefficients at most m base B digits long. Assume the primes satisfy $B < p < 2B$ and that arithmetic in $\mathbb{Z}_p$ costs $O(1)$. Estimate the time complexity of your algorithm in big O notation as a function of n, m and d. Make reasonable simplifying assumptions such as $n < B$ and $d < B$ as necessary. State your assumptions. Also helpful is

$$\ln n! < n \ln n \quad \text{for} \quad n > 1.$$

Question 4: A linear x-adic Newton iteration (15 marks).

Let p be an odd prime and let $a(x) = a_0 + a_1 x + ... + a_n x^n \in \mathbb{Z}_p[x]$ with $a_0 \neq 0$ and $a_n \neq 0$. Suppose $\sqrt{a_0} = \pm u_0 \bmod p$. The goal of this question is to design an x-adic Newton iteration algorithm that given u_0, determines if $u = \sqrt{a(x)} \in \mathbb{Z}_p[x]$ and if so computes u.

(a) Let

$$u = u_0 + u_1 x + ... + u_{k-1} x^{k-1} + u_k x^k + ...$$

Derive the Newton update formula for u_k. Show your working.

(b) Now test your update formula on the two polynomials $a_1(x)$ and $a_2(x)$ below using $p = 101$ and $u_0 = +5$. Please print out the sequence of values of $u_0, u_1, u_2, ...$ as you compute them. Note, one of the polynomials has a $\sqrt{}$ in $\mathbb{Z}_p[x]$, the other does not. So you will need to work out when the algorithm should stop lifting.
Do all calculations in Maple.

$$a_1 = 81\,x^6 + 16\,x^5 + 24\,x^4 + 89\,x^3 + 72\,x^2 + 41\,x + 25$$

$$a_2 = 81\,x^6 + 46\,x^5 + 34\,x^4 + 19\,x^3 + 72\,x^2 + 41\,x + 25$$

(c) The update formula requires $u_0 \neq 0$. Explain briefly what you should you do if $a_0 = 0$ and you want to compute $\sqrt{a(x)} \in \mathbb{Z}_p[x]$ if it exists.