MACM 202 Assignment 5, Fall 2005

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This assignment is worth 10% of your grade. It is due Tuesday November 22nd at 12 noon. A late penalty of 20% will apply for each day late. Attempt all questions.

Question 1 (20 marks)

1. Consider $T' = k(T_r - T)$. Solve the DE by hand using the method of separation of variables. Express your solution in terms of T(0).

2. Consider the Gompertz equation on page 163. Solve the DE by hand using the method of separation of variables. Try to express the solution in terms of x(0) in the form stated on page 163. Also, solve the DE using Maple and try to get Maple to express the solution in this form!

Question 2 (20 marks)

Consider the initial value problem $x'(t) = f(x(t))$ where $f(x) = x(3 - x) - 1$ and $x(0) = 2$. Compute the exact solution using `dsolve` and determine $x(2)$ to 10 decimal places.

Apply Euler's method, Heun's method and the second order Taylor series method to estimate $x(2)$ for $n = 2, 4, ..., 256$ steps, that is, for stepsizes $h = 1, 1/2, ..., 1/128$. Produce a table showing the estimates and the error of the estimates. To make the table automatically, use the `printf` command in a for loop to print each line of the table.

For each method, is the error proportional to h, h^2, h^3 ?

Question 3 (40 marks)

1. Let P be a population that lives in a closed community. Suppose a virus spreads through the population and suppose everyone is susceptible to the virus and that infected individuals carry the virus until they die. This assumption means that eventually everyone will become infected. Let $y(t)$ be the proportion of the population with the virus at time t. That is, $0 \leq y(t) \leq 1$ and the proportion of the population which does not have the virus yet is $1 - y(t)$. What differential equation would model the spread of the virus through the population? For the DE that you give, graph solution curves that illustrate the spread over time of the virus through the population for appropriate initial values.

Lake	V (km$^3 \times 10^{+3}$)	r (km^3/year)
Superior	12.2	65.3
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

2. Let $x(t)$ denote a fish population at time t. Consider models of the form $x'(t) = G(x) - H(x,t)$ where $G(x)$ models the natural growth of the fish population and $H(x,t)$ represents a *seasonal harvest* taken by fishermen. Construct such a model where $H(x,t) = xg(t)$ where $0 \le g(t) \le 1$. Graphically display what happens to the fish population for appropriate choices of $x(0)$ using the `DEplot` command.

3. Consider a lake of constant volume V. Let $Q(t)$ be the amount of pollutant in the lake at time t and let $c(t) = Q(t)/V$ be the concentration of pollutant at time t. Suppose pollutant enters the lake through rivers entering the lake and directly from factories bordering the lake. Suppose pollutant exits the lake via the outlet river. Assume also that the pollutant is always evenly distributed throughout the lake. The problem is to determine $c(t)$. Suppose factories add pollutant to the lake at a constant rate P. Let the concentration of pollutant entering the lake by rivers at rate r be k. Let the rate at which water leaves the lake also be r.

   (i) Find an expression for $c'(t)$. Let $c(0) = c_0$. Solve the differential equation for $c(t)$ using Maple. You should obtain

   $$c(t) = k + P/r + (c_0 - P/r - k)\,\mathrm{e}^{-\frac{r}{V}t}.$$

   (ii) If the addition of pollutants to the lake is terminated at some time t, i.e., $k \to 0$ and $P \to 0$, determine the time T that must elapse before the concentration $c(t)$ of pollutant is reduced to 10% of its original value.

   (iii) Using the given data, determine from part (b) the time T necessary to reduce the pollutant to 10% of the original value.

Question 4 (20 marks)

1. Do Excercise 5.12.

2. Consider $y'' = -4y' - ky$ with $y(0) = 1, y'(0) = 0$. By hand, solve the DE using the characteristic equation for $k = 3, 4, 8$. For the solution which is oscillatory, express your answer in the form $y(t) = e^{kt}[c_1 \sin(\omega t) + c_2 \cos(\omega t)]$ and in the form $y(t) = Ae^{kt} \sin(\omega t + \phi)$. Use Maple to check your answers.