

Assignment 5, MACM 204, Fall 2012

Due Monday December 3rd at 3:30pm at the beginning of class.

Late penalty: -20% for up to 48 hours late. 0 after that.

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Please attempt each question in a separate worksheet (so that you don't destroy your previous work). Print your Maple worksheets (you may print double sided if you wish) and hand them in to me.

There are 8 questions. Attempt all questions.

Question 1

Consider a chemical reaction involving chemicals A, B, and C in which A is converted to B at a rate k_1 and chemical B is converted to C at a rate of k_2 as illustrated in the compartment model below. I created the figure in Maple using the "Canvas" option under the insert menu, which I didn't know about but is very useful. Letting $A(t), B(t), C(t)$ be the amount of chemical A, B, C at time t we can model the chemical reactions with the differential equations

$$A'(t) = -k_1 \cdot A(t),$$

$$B'(t) = k_1 \cdot A(t) - k_2 \cdot B(t),$$

$$C'(t) = k_2 \cdot B(t).$$

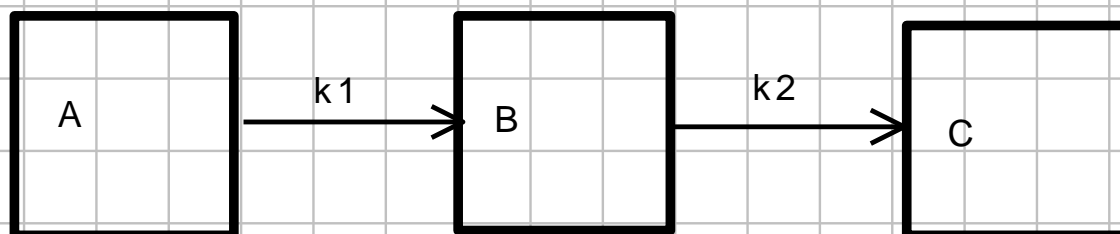
Part (a): Solve the differential equations using `dsolve` for the initial value $A(0) = N, B(0) = 0, C(0) = 0$. Here $N > 0$ is the initial amount of chemical A. Try to simplify the solutions as best you can.

You will see that there is a factor of $k_1 - k_2$ in the denominators which means the solutions are not valid for the case $k_1 = k_2$. So solve the differential equations for this case too. Again, try to simplify the solutions.

Part (b): Now, looking at the model, can you predict what $A(\infty), B(\infty), C(\infty)$ are? You don't need to do any math to figure this out. For the case $k_1 = k_2$ try also taking the limit of $A(t), B(t), C(t)$ as $t \rightarrow \infty$. You will need to tell Maple that $k_2 > 0$. You can do this using the assuming statement like this

> limit(A(t), t = infinity) assuming k2>0;

Part(c): For $N = 5, k_1 = 0.2, k_2 = 0.1$, solve the differential equations and graph $A(t), B(t), C(t)$ versus t on the same graph for a suitable domain using the plot command. Attach a legend = ["A", "B", "C"]. To do this you need to convert from a set of functions to a list of functions. I'll let you figure out how to do that.



► Solution 1

▼ Question 2

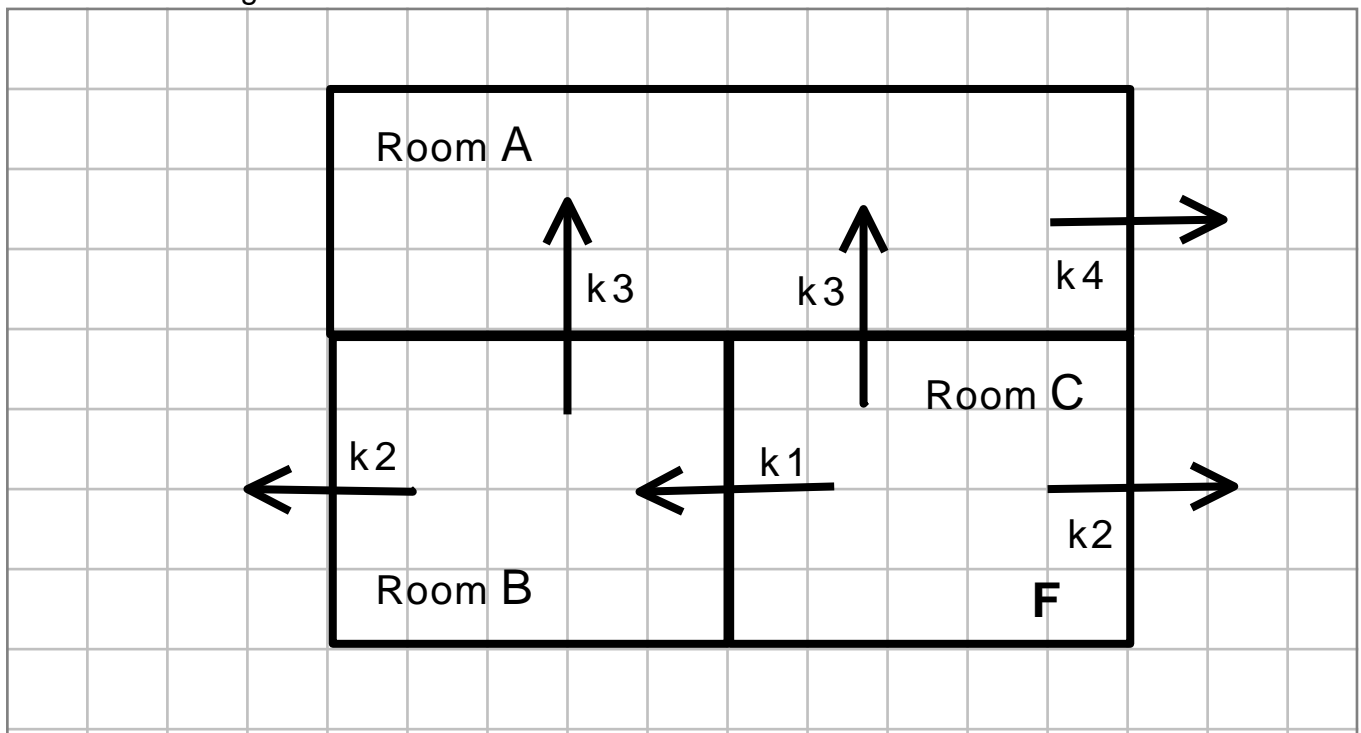
Shown in the figure below is a house with three rooms, A, B and C. Rooms B and C are the same size and shape. There is a furnace **F** in room C which heats room C. Let $A(t), B(t), C(t)$ be the temperature at time t in rooms A, B, C respectively and let A_m be the outside temperature. Shown in the figure are the cooling rate constants k_1, k_2, k_3, k_4 for how heat moves through the walls of the house.

Part (a) Write down a system of 3 differential equations $A'(t) = \dots, B'(t) = \dots, C'(t) = \dots$ for the house.

Part (b) Determine the temperature equilibrium point as a function of $F, A_m, k_1, k_2, k_3, k_4$. Try to simplify the formulas by writing them in the form $A_m + f(k_1, k_2, k_3, k_4) \cdot F$.

Part (c) For $F = 5, A_m = 0, k_1 = 0.3, k_2 = 0.1, k_3 = 0.3, k_4 = 0.2$ determine the temperature equilibrium, solve the differential equations using dsolve, and compute the limit of $A(t), B(t), C(t)$ as $t \rightarrow \infty$.

Part (d) Finally, graph $A(t), B(t), C(t)$ on the same graph on a suitable domain. Note, if the graph looks wrong this could be because your differential equations are wrong. Watch out for sign errors.



► Solution 2

▼ Question 3

Shown in the figure below is lake Erie and lake Ontario and the main rivers flowing through them (the arrows). Google says that the volume of lake Erie is about $500 \cdot km^3$ and lake Ontario is about $1500 \cdot km^3$ and the amount of water flowing through the lakes is about $60 \cdot km^3$ per year. Yes, that's kilometers cubed. The goal is to model the amount

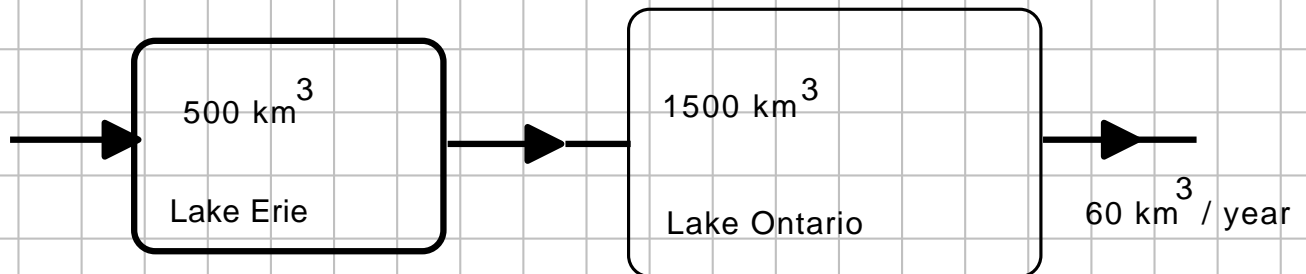
of pollution in the two lakes at time t (years). We will assume that initially, there is no pollution in either lake and that the river flowing into lake Erie is polluted and is bringing in 30 tons of pollutant per year.

Let $Er(t)$ be the amount of pollutant (in tons) in lake Erie at time t and let $On(t)$ be the amount of pollutant (in tons) in lake Ontario at time t (years).

Set up two differential equations, one for the amount of pollution in lake Erie at time t and the other for the amount of pollution in lake Ontario at time t . This problem is very much like the tank problem in the last assignment, except that here we have two tanks (two lakes).

Solve the differential equations and plot the solutions for a suitable time domain. You should see that the amount of pollutant in each lake increases from 0 to a maximum. What are the maximums?

Using the DEplot command in the DEtools package, generate a field plot with solution curves for initial values $Er(0) = 0, On(0) = 0$ and $Er(0) = 500, On(0) = 0$ and $Er(0) = 500, On(0) = 1000$ on the same plot.



► Solution 3

▼ Question 4

The Kermack-McKendrick virus spread model (where we partition the individuals in a population into those which are susceptible, infected, and recovered), is given by

$$S'(t) = -\beta \cdot S(t) \cdot I(t), \quad I'(t) = \beta \cdot S(t) \cdot I(t) - \alpha \cdot I(t), \quad R'(t) = \alpha \cdot I(t)$$

where $S(t), I(t), R(t)$ is the proportion of the population which is susceptible, infected and recovered at time t . We cannot solve the differential equations for nice formulas but we can solve them numerically for a given set of initial values.

For $\beta = 0.3, \alpha = 0.1, S(0) = 0.99, I(0) = 0.01, R(0) = 0.00$ solve the differential equations using `dsolve(..., numeric)` and plot the solutions for $S(t), I(t), R(t)$ on the same plot using the `odeplot` command in the `plots` package. You should see an epidemic, that is, $I(t)$ increases to a maximum and then drops down to 0. Please include a decent legend on the plot. Looking at the plot, what proportion of the population never becomes infected, i.e., what is $S(\infty)$? [Rough estimate from the plot is fine.]

Three years ago the H1N1 flu virus was hitting British Columbia. Perhaps you got vaccinated? I did. Dr. Perry Kendall, the provincial health officer was advising us that

we should all get a vaccination, not just to protect us individually, but to prevent H1N1 from becoming an epidemic. How can that work? The idea is that if a high enough proportion of the population gets the vaccination then they cannot contract the virus, so the virus cannot spread as easily through the population, and this can prevent an epidemic (growing). We can test this hypothesis in the SIR virus model. Suppose 50% of the population gets the H1N1 vaccine and 1% has the H1N1 virus (so 49% is susceptible). We can simulate this in the model using $S(0) = 0.49, I(0) = 0.01, R(0) = 0.50$. Solve the differential equations using this set of initial values to determine if this makes the virus endemic (die out) or not. If not, find, roughly, the %age of the population that must be vaccinated (this is for $\alpha=0.1, \beta=0.3$) to make the virus endemic experimentally.

► Solution 4

The following 4 questions review the material on the first four assignments (they will be marked). Basically you need practice using Maple to master it. For questions 6 and 8 you will need to rotate a vector by 60 degrees ($\pi/3$ radians). If you need it, the rotation matrix for rotating anti-clockwise by θ degrees/radians is

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

▼ Question 5

Do you remember Simpson's rule from MATH 152? It's a numerical integration rule. To approximate the definite integral $A = \int_a^b f(x) dx$ we let $h = \frac{(b-a)}{2}$ and compute $f(a), f(a+h), f(b = a+2 \cdot h)$. Simpson's rule fits (interpolates) the three points

$(a, f(a)), (a+h, f(a+h)), (a+2 \cdot h, f(a+2 \cdot h))$

with a quadratic $Q(x) = A \cdot x^2 + B \cdot x + C$. Then it approximates the definite integral A with

$S = \int_a^b Q(x) dx$ thus obtaining the formula for Simpson's rule namely

$$S = \frac{h}{3} \cdot (f(a) + 4 \cdot f(a+h) + f(a+2 \cdot h)).$$

Note this approximation is much more accurate than the Trapezoidal rule

$$T = \frac{h}{2} \cdot (f(a) + 2 \cdot f(a+h) + f(a+2 \cdot h)).$$

Use Maple to derive the formula for Simpson's rule.

Step 1: Find $Q(x)$ using data values $(a, f(a)), (a+h, f(a+h)), (a+2 \cdot h, f(a+2 \cdot h))$. Do this by setting up a linear system of equations to solve for A, B, C .

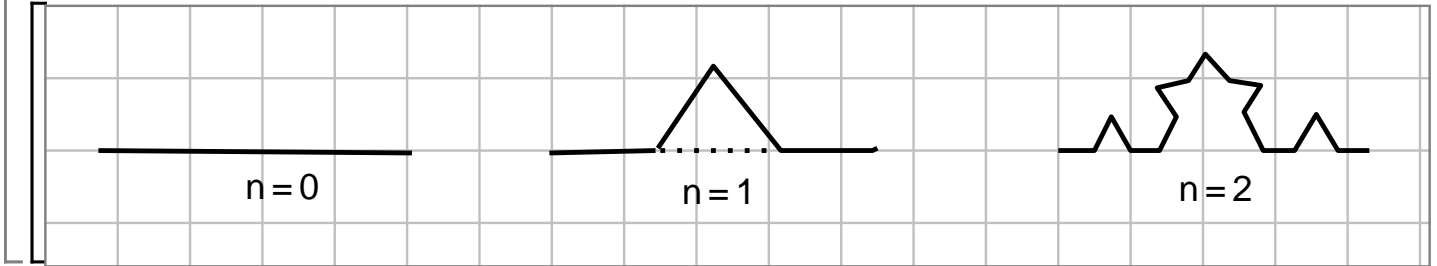
Step 2: Once you have a formula for $Q(x)$ just use Maple to compute $S := \int_a^b Q(x) dx$. To

get Simpson's formula you'll need to simplify the result. If you are not sure what is going on, sketch a function $f(x)$, mark the points $x = a, a+h, b$ and $f(a), f(a+h), f(b)$, then sketch $Q(x)$ and shade the area S . Then try to come up with $Q(x)$ by hand and use Maple to solve the linear system.

► Solution 5

▼ Question 6

The Koch snowflake (look for a picture using google) can be constructed as follows. Start with an equilateral triangle ABC. For each edge in the triangle (shown below as $n=0$ is one of the edges) we will build a fractal as follows: Replace the edge by the four edges shown in the figure for $n=1$, i.e., divide the edge into three equal subintervals and form a small equilateral triangle from the middle edge but omit the base (the dotted line). Repeat this procedure n times for each new edge. The result for $n=2$ is shown below. Write a recursive Maple program to do this.



► Solution 6

▼ Question 7

The Thue Morse sequence $T(n)$ is defined for $n = 0, 1, 2, 3, 4, \dots$ as follows: Let B_n be the number of 1 bits in the binary representation of n . Then

$$T(n) = \begin{cases} 0 & \text{if } B_n \text{ is even} \\ 1 & \text{if } B_n \text{ is odd} \end{cases}$$

For example, the binary representation of the integer 13 is 1101 because $13 = 8+4+1$. Hence $B_{13} = 3$ thus $T(13) = 1$.

One way to compute $T(n)$ would be to convert n to binary and count the number of 1 bits. But notice that $T(2 \cdot k) = T(k)$ and $T(2 \cdot k + 1) = T(k) + 1 \pmod{2}$. This suggests a simple recursive method. To compute the integer quotient and integer remainder of $n \div 2$ in Maple use

```
> iquo(13,2);
```

6 (13.1)

```
> irem(13,2);
```

1 (13.2)

Write a Maple procedure to compute $T(n)$. Compute the Thue Morse sequence for $n = 0, 1, 2, \dots, 40$. Check that your result agrees with this

```
> cat( seq(T(n),n=0..40) );
```

011010011001011010010110011001100110011001100 (13.3)

► Solution 7

Question 8

Let V be the vector $[1,0]$ and set $S_0 := [0, 0]$ to be the origin.

To construct the Koch Snowflake do the following:

for $i = 1, 2, 3, \dots, n$ **do**

 if $T(i) = 0$ then take a unit step from the current point S_k in the direction of V
 otherwise rotate the direction V by 60 degrees.

Here $T(i)$ is the Thue-Morse sequence from question 7. This will generate a sequence of points S_0, S_1, S_2, \dots . For $n = 10,000$ generate a plot of the points where you join the points by straight line segments. This should converge in a strange way to the Koch snowflake!! If you plot the first 1000 points only you will see why it is strange. Very strange.

► Solution 8