Assignment 5, MACM 204, Fall 2014

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Due Tuesday November 25th at 4:00pm.
Late penalty: -20% for up to 46 hours late. 0 after that.

There are 9 questions. Questions 5-9 are review questions to prepare you for the final.
Attempt all questions.

Question 1

Consider a chemical reaction involving chemicals A, B, and C in which A is converted to B at a rate k_1 and chemical B is converted to C at a rate of k_2 as illustrated in the compartment model below. I created the figure in Maple using the "Canvas" option under the insert menu, which I didn't know about but is very useful. Letting $A(t), B(t), C(t)$ be the amount of chemical A, B, C at time t we can model the chemical reactions with the differential equations

$$A'(t) = -k_1 \cdot A(t),$$
$$B'(t) = k_1 \cdot A(t) - k_2 \cdot B(t),$$
$$C'(t) = k_2 \cdot B(t).$$

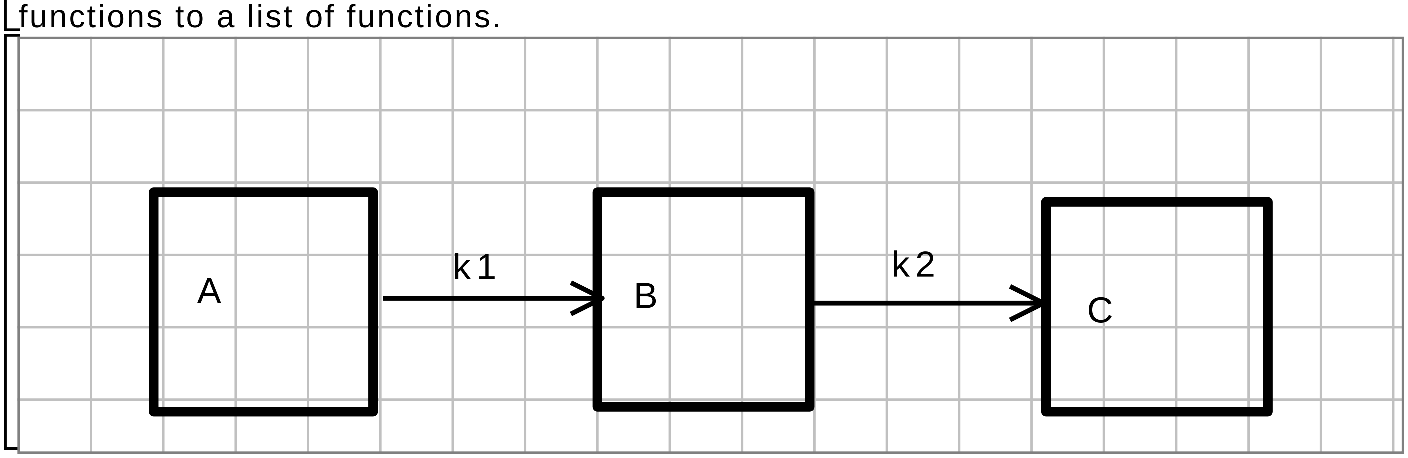
Part (a): Solve the differential equations using `dsolve` for the initial value $A(0) = N, B(0) = 0, C(0) = 0$. Here $N > 0$ is the initial amount of chemical A. Try to simplify the solutions as best you can.

You will see that there is a factor of $k_1 - k_2$ in the denominators which means the solutions are not valid for the case $k_1 = k_2$. So solve the differential equations for this case too. Again, try to simplify the solutions.

Part (b): Now, looking at the model, can you predict what $A(\infty), B(\infty), C(\infty)$ are? You don't need to do any math to figure this out. For the case $k_1 = k_2$ try also taking the limit of $A(t), B(t), C(t)$ as $t \to \infty$. You will need to tell Maple that $k_2 > 0$. You can do this using the assuming statement like this

```
> limit( A(t), t = infinity ) assuming k2>0;
```

Part(c): For $N = 5, k_1 = 0.2, k_2 = 0.1$, solve the differential equations and graph $A(t), B(t), C(t)$ verses t on the same graph for a suitable domain using the plot command. Attach a legend = ["A","B","C"]. To do this you need to convert from a set of functions to a list of functions.



Question 2

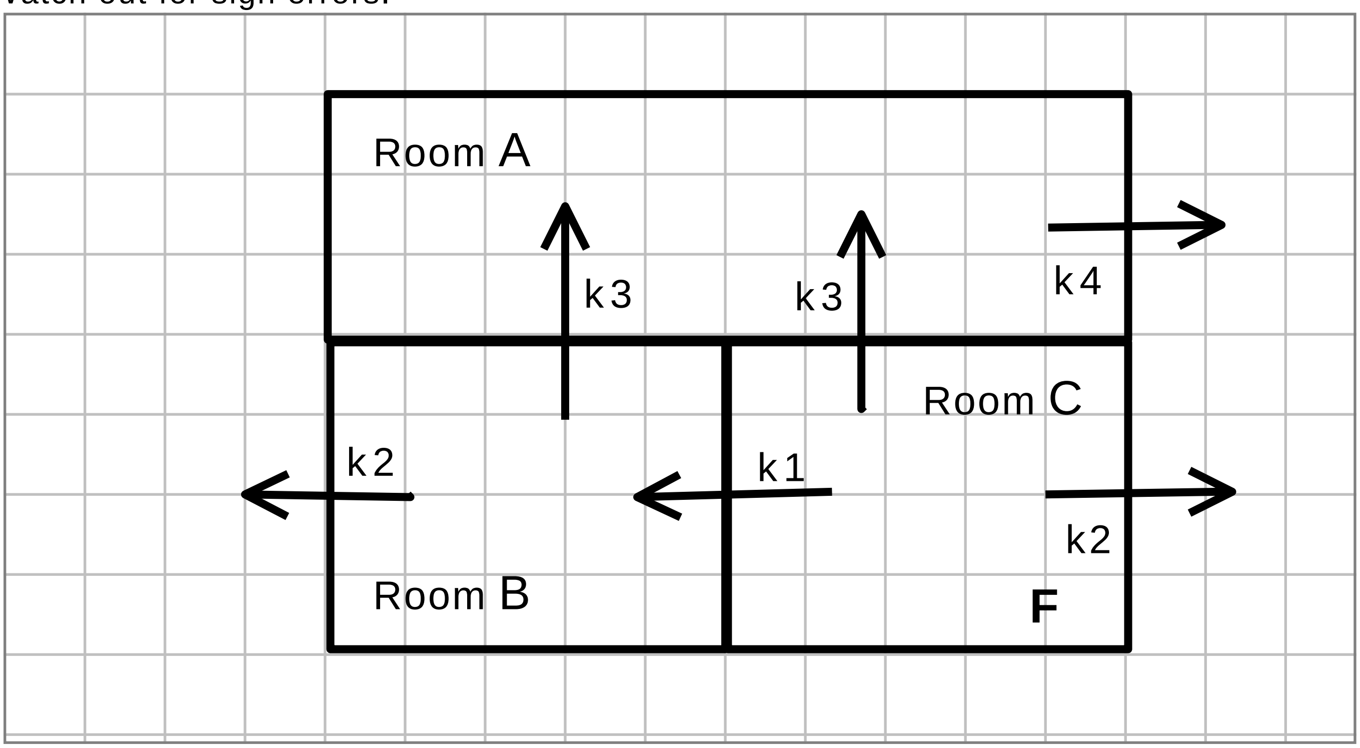
Shown in the figure below is a house with three rooms, A, B and C. Rooms B and C are the same size and shape. There is a furnace **F** in room C which heats room C. Let $A(t)$, $B(t)$, $C(t)$ be the temperature at time t in rooms A, B, C respectively and let Am be the outside temperature. Shown in the figure are the cooling rate constants k_1, k_2, k_3, , k_4 for how heat moves through the walls of the house.

Part (a) Write down a system of 3 differential equations $A'(t) = ...,\ B'(t) = ...,\ C'(t) = ...$ for the house.

Part (b) Determine the temperature equilibrium point as a function of F, Am, $k1$, $k2$, $k3$, $k4$. Try to simplify the formulas by writing them in the form $Am + f(k1, k2, k3, k4) \cdot F$.

Part (c) For $F = 5,\ Am = 0,\ k1 = 0.3,\ k2 = 0.1,\ k3 = 0.3,\ k4 = 0.2$ determine the temperature equilibrium, solve the differential equations using dsolve, and compute the limit of $A(t)$, $B(t)$, $C(t)$ as $t \to \infty$.

Part (d) Finally, graph $A(t)$, $B(t)$, $C(t)$ on the same graph on a suitable domain. Note, if the graph looks wrong this could be because your differential equations are wrong. Watch out for sign errors.



Question 3

Shown in the figure below is lake Erie and lake Ontario and the main rivers flowing through them (the arrows). Google says that the volume of lake Erie is about $500 \cdot km^3$ and lake Ontario is about $1500 \cdot km^3$ and the amount of water flowing through the lakes is about $60 \cdot km^3$ per year. Yes, that's kilometers cubed. The goal is to model the amount of pollution in the two lakes at time t (years). We will assume that initially, there is no pollution in either lake and that the river flowing into lake Erie is polluted and is bringing in 30 tons of pollutant per year.

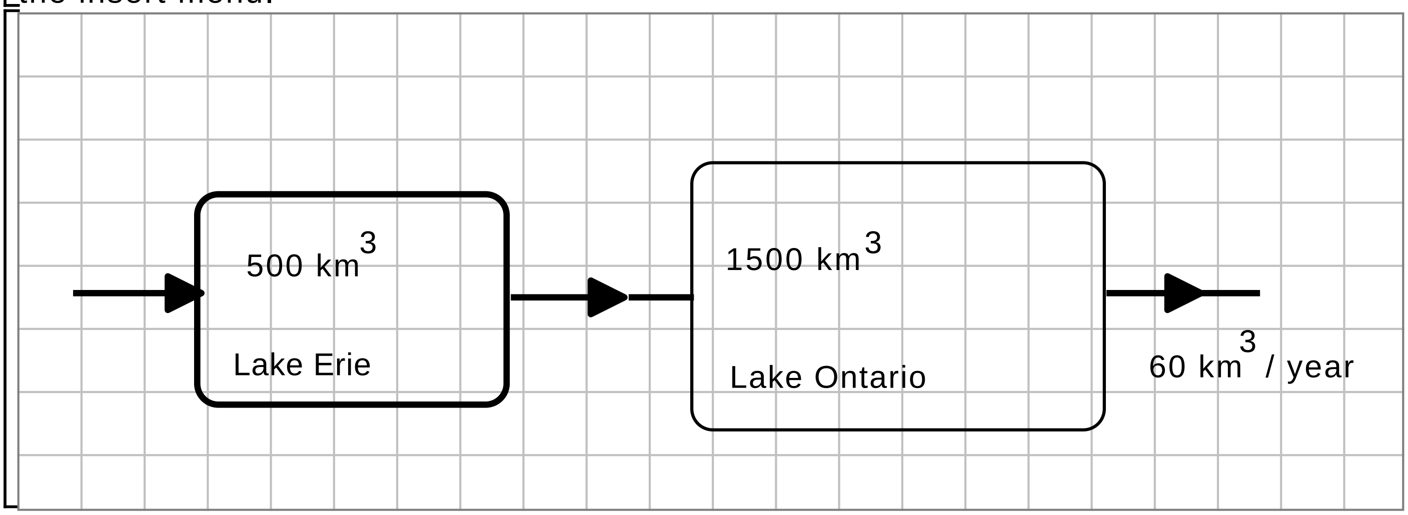
Let $Er(t)$ be the amount of pollutant (in tons) in lake Erie at time t and let $On(t)$ be the amount of pollutant (in tons) in lake Ontario at time t (years).

Part (a) Set up two differential equations, one for the amount of pollution in lake Erie at time t and the other for the amount of pollution in lake Ontario at time t. This problem is very much like the tank problem in the last assignment, except that here we have two tanks (two lakes).

Parb (b) Solve the differential equations and plot the solutions for a suitable time domain. You should see that the amount of pollutant in each lake increases from 0 to a maximum. What are the maximums?

Part (c) Using the DEplot command in the DEtools package, generate a field plot with solution curves for initial values $Er(0) = 0, On(0) = 0$ and $Er(0) = 500, On(0) = 0$ and $Er(0) = 500, On(0) = 1000$ on the same plot.

Part (d) Reproduce the figure below in Maple. It's a "Canvas" which you can create from the insert menu.



Question 4

The Kermack-McKendrick virus spread model (where we partition the individuals in a population into those which are susceptible, infected, and recovered), is given by
$S'(t) = -\beta \cdot S(t) \cdot I(t),\ I'(t) = \beta \cdot S(t) \cdot I(t) - \alpha \cdot I(t),\ R'(t) = \alpha \cdot I(t)$
where $S(t), I(t), R(t)$ is the proportion of the population which is susceptible, infected and recovered at time t. We cannot solve the differential equations for nice formulas but we can solve then numerically for a given set of initial values.

Part (a) For $\beta = 0.3, \alpha = 0.1, S(0) = 0.99, I(0) = 0.01, R(0) = 0.00$ solve the differential equations using dsolve(..., numeric) and plot the solutions for $S(t), I(t), R(t)$ on the same plot using the `odeplot` command in the plots package. You should see an epidemic, that is, $I(t)$ increases to a maximum and then drops down to 0. Please include a decent legend on the plot. Looking at the plot, what proportion of the population never becomes infected, i.e., what is $S(\infty)$? [Rough estimate from the plot is fine.]

Five years ago the H1N1 flu virus hit British Columbia. Perhaps you got vaccinated? I did. Dr. Perry Kendall was the provincial health officer then. He was advising us that we should all get a vaccination, not just to protect us individually, but to prevent H1N1 from becoming an epidemic. How can that work? The idea is that if a high enough proportion of the population gets the vaccination then they cannot contract the virus, so the virus cannot spread as easily through the population, and this can prevent an epidemic (outbreak). We can test this hypothesis in the SIR virus model. Suppose 50% of the population gets the H1N1 vaccine and 1% has the H1N1 virus (so 49% is susceptible). We can simulate this in the model using

$$S(0) = 0.49, I(0) = 0.01, R(0) = 0.50.$$

Parb (b) Solve the differential equations using this set of initial values to determine if this makes the virus endemic (die out) or not. If not, find, roughly, the %age of the population that must be vaccinated (this is for $\alpha=0.1$, $\beta=0.3$) to make the virus endemic.

Question 5

Let $M(t)$ be the amount owed on a 30 year mortgage of $200,000 at time *t* years.
Suppose the annual interest rate on the mortgage is *r*=4%.
Suppose the term of the mortgage is 30 years, i.e., M(30) should be 0.
Suppose we pay $*P* per year. We can use the following differential equation model interest charges and our payments.

$M'(t) = r \cdot M(t) - P$ dollars per year

If we assume the interest is paid continuously (banks usually charge interest daily which is approximately continuous) and if we assume we make the payments continuously (banks usually require us to pay monthly or weekly which is approximately continuous over 30 years) then we can estimate the annual payment P as follows.

```
> de := diff( M(t),t) = r*M(t)-P;
  r := 0.04;
```

$$de := \frac{d}{dt} M(t) = r\, M(t) - P$$

$$r := 0.04$$

```
> sol := dsolve( {de, M(0)=200000}, M(t) );
```

$$sol := M(t) = 25\,P + e^{\frac{1}{25}t}\,(200000 - 25\,P)$$

To determine P we solve $M(30) = 0$ for P.

```
> eq := eval( rhs(sol), t=30.0 ) = 0;
```

$$eq := -58.00292308\,P + 6.640233846\,10^{5} = 0$$

```
> annualpayment := solve( eq, P );
```

$$annualpayment := 11448.10208$$

So we pay $11,448.10 dollars per year. Also, the total interest we pay is quite a bit

```
> interestpaid := 30*annualpayment - 200000;
```

$143,443.06

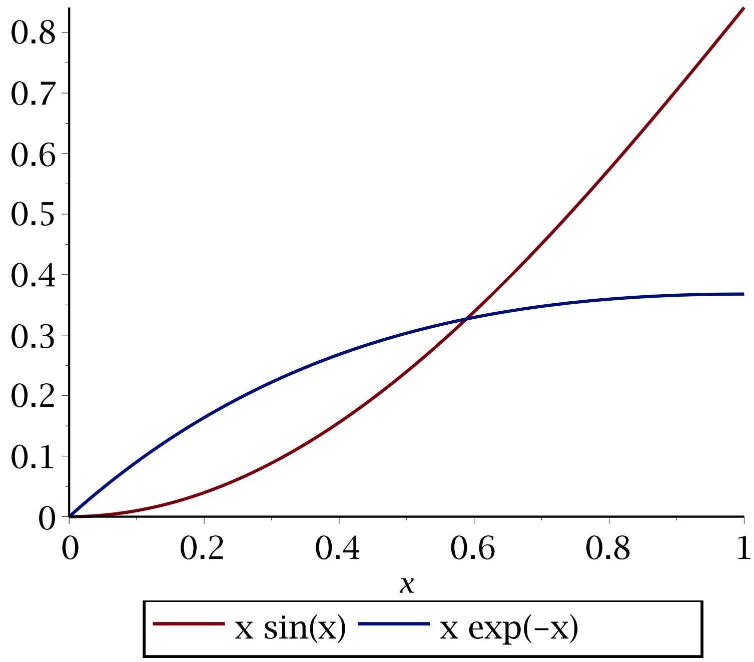
Suppose you want to pay off the mortgage faster to avoid paying all that interest!!
Suppose you decide to pay P = $15,000 pear year instead.

Part (a) Solve the differential equation with r = 4% per year and P = 15,000 per year and determine (i) how long it will take you to pay off the mortgage and (ii) how much interest you will pay.

Part (b) Now graph (**on the same plot**) *M(t)* for *P* = $11,448.10 **and** *M(t)* with *P* = $15,000 on a suitable domain. Include a legend and title.

Question 6

Generate the plot below of the two functions $f(x) = x \cdot \sin(x)$ and $g(x) = x \cdot e^{-x}$ on the domain $0 \le x \le 1$. Calculate the area A enclosed between the two curves as a definite integral. You will run into a difficulty when you try to solve $f(x) = g(x)$ using the solve command or if you try to do it by hand. I leave it up to you to decide what to do.



Question 7

Consider an equilateral triangle A B C.
The **Fractal game** is to do the following

Pick an initial point x_0 anywhere in the triangle.
for *k* from 1 to *n* do
Toss a die
If you get 1 or 2 set x_k to be half way from x_{k-1} to A.
If you get 3 or 4 set x_k to be half way from x_{k-1} to B.
Otherwise set x_k to be half way from x_{k-1} to C.

This generates a sequence on n points $x_1, x_2, ..., x_n$ which, not, must all be inside the triangle.

Program this in Maple and graph the points for n at least 5000. What picture do you get?

Note, to simulate a die use

```
> die := rand(1..6):
```

Then die(); will generate a random integer on [1,6]. E.g.

```
> die();
                                   4
> die();
                                   3
> die();
                                   6
```

Note, you can use Maple lists $[x, y]$ to represent the points A, B, C and the x_k .

Question 8

Consider the function $f(x, y)$ below

```
> f := x^4+y^4-2*x*y^3-2*y*x^3+x*y-x-y-8;
```

$$f := x^4 - 2x^3y - 2xy^3 + y^4 + xy - x - y - 8$$

It has one real critical point. To find it we can solve the system $f_x(x, y) = 0, f_y(x, y) = 0$. We will have to solve two simultaneous cubic equations

```
> sys := [diff(f,x)=0,diff(f,y)=0];
```

$$sys := [4x^3 - 6x^2y - 2y^3 + y - 1 = 0, -2x^3 - 6xy^2 + 4y^3 + x - 1 = 0]$$

Part (a) First, graph the curves $f_x = 0$ and $f_y = 0$ using the `plots[implicitplot]` command so you can see roughly where the real solution is.

Part (b) Next solve the equations using `fsolve` to find the critical point $x = a, y = b$.

Part (c) Check that it is correct by graphing the surface $f(x, y)$ and drawing a vertical line at $x = a, y = b$ through the surface on a suitable domain. Can you tell what kind of critical point it is?

Part (d) Finally, find an exact formula for the critical point using solve. Use the option

```
> _EnvExplicit := true;
```

to get exact formulas for the solutions. All except one will be complex. Make sure you input the function $f(x, y)$ correctly!

Question 9

Below is a Maple procedure `TrapezoidalRule(f,a,b,n)` that uses the Trapezoidal rule to approximate an integral of $f(x)$ on the interval $[a, b]$ with n trapezoids of width $h = (b - a)/n$. Note the input f is a procedure not a formula so that we can write `f(a)` instead `eval(f,x=a)`. The formula for the Trapezoidal rule is

$$T_n = \frac{h}{2} \cdot [f(a) + 2 \cdot f(a + h) + 2 \cdot f(a + 2 \cdot h) + \ldots + 2 \cdot f(a + (n - 1) \cdot h) + f(b)]$$

```
> TrapezoidalRule := proc(f::procedure,a,b,n::posint)
  local h,Tn,i;
    h := evalf((b-a)/n);
    Tn := evalf( f(a)+f(b) );
    for i to n-1 do Tn := Tn + 2 * evalf( f(a+i*h) ); od;
```

```
    h/2*Tn;
  end:
```

Here is the procedure applied to compute $\int_0^\pi x \cdot \sin(x)\, dx$ for $n = 8, 16, 32, 64$ and to print out the approximations and the error in the approximations.

```
> f := x -> x*sin(x); # input is a procedure
  a,b := 0,Pi;
  A := int(f(x),x=a..b); # this integral = Pi
  for n in [4,8,16,32,64] do
     T := TrapezoidalRule(f,a,b,n);
     E := abs( evalf(A-T) );
     printf(" n=%3d  Tn=%12.8f  error=%12.8f\n", n, T, E  );
  od:
```

$$f := x \rightarrow x \sin(x)$$

$$a, b := 0, \pi$$

$$A := \pi$$

```
 n=  4  Tn=  2.97841660  error=  0.16317605
 n=  8  Tn=  3.10111575  error=  0.04047691
 n= 16  Tn=  3.13149297  error=  0.01009968
 n= 32  Tn=  3.13906895  error=  0.00252370
 n= 64  Tn=  3.14096180  error=  0.00063085
```

Find the formula for Simpson's rule. You can use google or dig up one of your old Calculus texts.
Write a Maple procedure `SimpsonsRule(f,a,b,n)` that applies Simpson's rule to approximate the integral. Assume n is even.

Run your procedure for n=4,8,16,32 and output the same data. You should find that Simpson's rule is MUCH more accurate than the Trapezoidal rule. Notice that the error in the Trapezoidal rule appears to decrease by a factor of 4 each time we double n. What factor of decrease do you get for Simpson's rule?

Notice that because the value of the integal is π we can use Simpson's rule to compute π. How big does n need to be to get say 12 digits of accuracy for π?