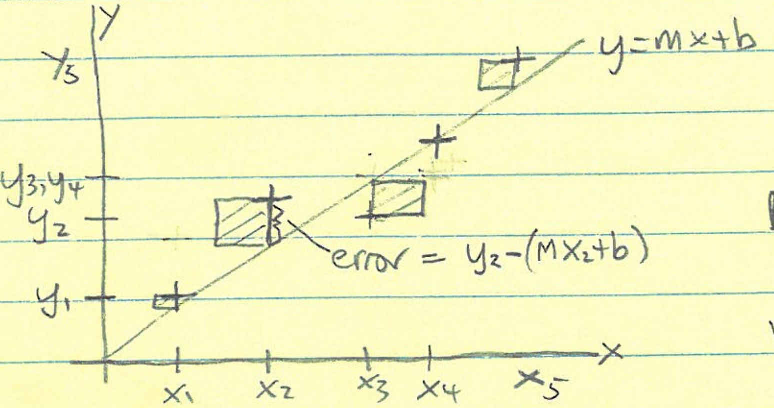
Lec 7

The method of Least Squares

20m

Suppose we are given n data points $(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$ obtained from an experiment. Suppose from Theory $y = mx+b$ for some constants m and b. The method of least squares chooses m,b to minimize the "square error"



$$E(m,b) = \sum_{i=1}^{n}\Big[\underbrace{y_i-(mx_i+b)}_{\text{error}}\Big]^2$$

Visually $E(m,b)$ is the area of the squares.

If $n \geq 2$ and $x_1 \neq x_2 \neq \cdots \neq x_n$ then there is a unique minimum.

To minimize $E(m,b)$ we must have $\underbrace{E_m(m,b)=0 \text{ and } E_b(m,b)=0}_{\text{local min is a critical point}}$.

$$E_m(m,b) = \sum_{i=1}^{n}\big[2(y_i-(mx_i+b))\big](-x_i) = 2\sum_{i=1}^{n}\big[+mx_i^2+bx_i-x_iy_i\big]$$

$$E_m(m,b)=0 \Rightarrow m\sum x_i^2 + b\sum x_i = \sum x_iy_i \qquad (1)$$

$$E_b(m,b) = \sum_{i=1}^{n} 2\big(y_i-(mx_i+b)\big)(-1) = 2\sum_{i=1}^{n}\big[mx_i+b-y_i\big]$$

$$E_b(m,b)=0 \Rightarrow m\sum x_i + \underbrace{\sum_{i=1}^{n} b}_{nb} = \sum y_i \qquad (2)$$

We have
$$\boxed{\begin{aligned}(\textstyle\sum x_i^2)m + (\textstyle\sum x_i)b &= \textstyle\sum x_iy_i \quad (1)\\ (\textstyle\sum x_i)m + nb &= \textstyle\sum y_i \quad (2)\end{aligned}}$$

[The sums $\sum x_i^2$, $\sum x_i$, $\sum x_iy_i$, $\sum y_i$ are numbers.]

[This is a linear system of equations in m and b.]

To show that the solution is a local minimum we can show $E_{mm}(m,b) > 0$ and $E_{bb}(m,b) > 0$ (easy) and $D(m,b) > 0$ (hard). [Assume $x_1 < x_2 < x_3 < \cdots < x_n$]