Lec22B Example Michael Monagan

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$K=\mathbb{Q}$, $\mathbb{Q}(x)$ $\quad F=\mathbb{Q}(i)$ $\quad L=\mathbb{Q}$

Example $\displaystyle\int \frac{1}{x^3+x} = \ln x - \tfrac{1}{2}\ln(x-i) - \tfrac{1}{2}\ln(x+i) = 1\cdot\ln x - \tfrac{1}{2}\ln(x^2+1).$

$$\frac{1}{x^3+x} = \frac{1}{x(x-i)(x+i)} = \frac{1}{x} + \frac{-\frac{1}{2}}{x-i} + \frac{-\frac{1}{2}}{x+i}$$

$R(z) = \operatorname{res}(C - zD', D, x)$ $\quad C=1,\ D = x^3+x,\ D' = 3x^2+1.$

$= \operatorname{res}(1 - z(3x^2+1), x^3+x, x)$

$= \operatorname{res}(\underbrace{(-3z)x^2 + (1-z)}_{A}, \underbrace{x^3+x}_{B}, x)$ $\quad 0, i, -i$

$= (1-z)(1+2z)^2$

$m = \deg A \quad n = \deg B, \quad B(\beta_j)=0$

$\operatorname{res}(A,B) = (-1)^{mn}\cdot \operatorname{lc}(B)^m \cdot \prod_{j=1}^{n} A(\beta_j)$

$= (-1)^{2\cdot 3}\cdot 1^2 \cdot A(0)\cdot A(i)\cdot A(-i)$

$= 1\cdot(1-z)(1+2z)(1+2z)$

The distinct roots of $R(z)$ are $1, -\frac{1}{2}$. $\Rightarrow \alpha_1 = 1, \alpha_2 = -\frac{1}{2}$.

Then $\quad v_i = \gcd(C - \alpha_i D', D)$

$v_1 = \gcd(1 - 1\cdot(3x^2+1), x^3+x) = \gcd(-3x^2, x^3+x) = \underline{x}.$

$v_2 = \gcd(1 - (-\frac{1}{2})(3x^2+1), x^3+x)$

$= \gcd(\frac{3}{2} + \frac{3}{2}x^2, x^3+x)$

$= \gcd(3/2(1+x^2), x(1+x^2)) = 1\cdot x^2+1.$

Thus $\displaystyle\int \frac{dx}{x^3+x} = 1\cdot\ln(x) + -\tfrac{1}{2}\ln(x^2+1).$

So $\int \frac{C}{D}$ depends on the roots of $R(z) = \operatorname{res}(C - zD', D, x)$ and not on the roots of $D(x)$. $R(z)$ is called the Trager-Rothstein resultant. Can we compute $R(z)$ without factoring $D(x) = (x-\beta_1)(x-\beta_2)\cdots$?

Yes, use the Euclidean algorithm or the det of Sylvester's matrix.

Let $A = a_m x^m + \cdots + a_0$ and $B = b_n x^n + \cdots + b_0$.
Sylvester's matrix is the $m+n$ by $m+n$ matrix

$$S = \begin{bmatrix} a_m & a_{m-1} & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & a_m & a_{m-1} & \cdots & a_0 & \cdots & 0 \\ & & \ddots & & & & \\ 0 & \cdots & 0 & a_m & a_{m-1} & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & b_n & \cdots & & b_0 & 0 \cdots & 0 \\ & & \ddots & & & & \\ 0 & \cdots & 0 & b_n & - & - & b_0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ times} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} m \text{ times.} \end{matrix}$$

Theorem: $\operatorname{res}(A, B) = \det(S)$.

$C = 1,\ D = x^3 + x \quad \operatorname{res}(C - zD', D) = \operatorname{res}(\underbrace{(-3z)x^2 + (1-z)}_{A}, \underbrace{x^3 + x}_{B}, \underline{\underline{x}})$

$$S = \begin{bmatrix} -3z & 0 & 1-z & 0 & 0 \\ 0 & -3z & 0 & 1-z & 0 \\ 0 & 0 & -3z & 0 & 1-z \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \qquad S_{3,5}$$

$\mathbb{Z}[z]$

$$\begin{aligned} \det(S) &= (1-z)\cdot(-1)^{3+5}\cdot\det(M.) \\ &= (1-z)\cdot 1\cdot(1+2z)^2 \\ &= R(z). \end{aligned}$$

$$M = \begin{bmatrix} -3z & 0 & 1-z & 0 \\ 0 & -3z & 0 & 1-z \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} C_1 \leftarrow C_1 - C_3 \\ C_2 \leftarrow C_2 - C_4 \end{matrix} \begin{bmatrix} -1-2z & 0 & 1-z & 0 \\ 0 & -1-2z & 0 & 1-z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = N$$

$\det(M) = \det(N) = (-1-2z)^2 = (1+2z)^2$