Lec23B Differentating Logarithmic and Exponential Polynomials Michael Monagan

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Theorem 12.2 (differentiation of logarithmic polynomials)

Let F be a differential field and $F(\theta)$ be a logarithmic transcendental differential extension of F with $\theta' \neq 0$.
I.e. $\theta = \log(u)$ for some $u \in F$, $u' \neq 0$, $\theta \notin F$.

If $a = a_n\theta^n + \cdots + a_1\theta + a_0 \in F[\theta]$ with $n > 0$, $a_n \neq 0$

(i) $a' = \frac{d}{dx} a(\theta) \in F[\theta]$
(ii) if $a_n' = 0$ then $\deg_\theta a' = n-1$
(iii) if $a_n' \neq 0$ then $\deg_\theta a' = n$

Theorem 12.3 (differentiation of exponential polynomials)

Let F be a differential field and $F(\theta)$ be an exponential transcendental differential extension of F with $\theta' \neq 0$.
I.e. $\theta = e^u$ for some $u \in F$, $u' \neq 0$, $\theta \notin F$

(i) If $a = a_n\theta^n + \cdots + a_1\theta + a_0 \in F[\theta]$ with $n > 0$, $a_n \neq 0$

$$a' = \frac{d}{dx} a(\theta) \in F[\theta] \text{ and } \deg_\theta a' = n$$

(ii) If $h \in F\backslash\{0\}$ and $m \in \mathbb{Z}\backslash\{0\}$ Then

$$(h\theta^m)' = \bar{h}\theta^m \text{ for some } \bar{h} \in F\backslash\{0\}$$

(iii) $a(\theta) \mid a'(\theta) \Rightarrow a(\theta) = h\theta^m$ for some $h \in F$, $m \in \mathbb{Z}$

Examples. $F[\theta] = \mathbb{Q}(x)[\log x]$

$$a = x\log^2 x + 2\log x = x\theta^2 + 2\theta \in F[\theta]$$

$$a' = 1\cdot\log^2 x + x\cdot 2\log x\cdot\frac{1}{x} + 2/x$$

$$= \log^2 x + 2\log x + \frac{2}{x} = \theta^2 + 2\theta + \frac{2}{x} \in \underline{F[\theta]}$$

$$a = 1\cdot\log^2 x + x\log x = \theta^2 + x\theta \in F[\theta].$$

$$a' = 2\log x\cdot\frac{1}{x} + 1\cdot\log x + x\cdot\frac{1}{x}$$

$$= \left(\frac{2}{x}+1\right)\log x + 1 = \left(\frac{2}{x}+1\right)\theta^1 + 1 \in F[\theta].$$

Prop n 12.2. $\theta = \log u$, $u \in F$, $\boxed{\theta \notin F.}$ Note $\theta' = \frac{u'}{u} \in F$

$$a = a_n\theta^n + a_{n-1}\theta^{n-1} + \cdots + a_1\theta + a_0 \quad \text{where } a_i \in F,\ n > 0.$$

$$a' = a_n'\theta^n + na_n\theta^{n-1}\theta' + a_{n-1}'\theta^{n-1} + \cdots + a_1'\theta + a_1\theta' + a_0'$$

$$= \underset{F}{\underset{\Uparrow}{a_n'}}\theta^n + (\underset{F}{\underset{\Uparrow}{\underline{na_n\theta'}}} + \underset{F}{\underset{\Uparrow}{a_{n-1}'}})\underline{\theta}^{n-1} + \cdots + (\underset{F}{\underset{\Uparrow}{a_1}}\underset{F}{\underset{\Uparrow}{\theta'}} + \underset{F}{\underset{\Uparrow}{a_0}}) \in F[\theta].$$

CASE $a_n' \neq 0$ then $a' = a_n'\theta^n + \cdots \Rightarrow \deg_\theta a' = n.$
(a_n is a constant)

CASE $a_n' = 0 \Rightarrow \deg_\theta(a') \leq n-1.$ Can this $< n-1$?

$\underline{\text{TAC}}$. Suppose $na_n\theta' + a_{n-1}' = 0$

$$\Rightarrow \int(na_n\theta' + \underline{a_{n-1}'})\,dx = k \quad \text{a constant.}$$

$$\Rightarrow na_n\theta + a_{n-1} = k.$$

$$\Rightarrow \theta = (\underset{F}{\underset{\Uparrow}{k}} - \underset{F}{\underset{\Uparrow}{a_{n-1}}})/(\underset{F}{\underset{\Uparrow}{n}}\underset{F}{\underset{\Uparrow}{a_n}}) \in F \quad \text{contradicting } \theta \notin F.$$