Lec24C Logarithmic Case: Polynomial Part Michael Monagan

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The Polynomial Part $\int P$, $P \in F[\theta]$, $\theta = \log u$

$\int \left(x^2 \log^2 x + \log x + \frac{1}{x}\right) dx$

$\mathbb{Q}(x)(\log x)$

Let $P = p_\ell \theta^\ell + \cdots + p_1 \theta + p_0$ where $p_i \in F$.

Liouville's theorem says if $\int P$ is elementary then

$$(*) \quad \int P = \underset{\in F(\theta)}{V_0(\theta)} + \sum \underset{\in \bar{K}}{c_i} \log \underset{\in F(\theta)}{v_i(\theta)} \qquad \text{WLOG } v_i \in F[\theta]$$

$\log \frac{A}{B} = \log A - \log B$

Let $V_0(\theta) = \frac{a(\theta)}{b(\theta)}$ where $\gcd(a,b) = 1$ and $\mathrm{lc}_\theta\, b = 1$. in $F[\theta]$

WLOG assume $V_i(\theta)$ are monic, irreducible in $F[\theta]$

$\log(AB) = \log(A) + \log(B)$

$$(*)' \quad \Rightarrow \quad P = p_\ell \theta^\ell + \cdots + p_0 = \frac{a'(\theta)}{b(\theta)} - \frac{b'(\theta)a(\theta)}{b(\theta)^2} + \sum c_i \frac{V_i(\theta)'}{V_i(\theta)}$$

If $\deg_\theta b > 0$ the terms in the PFD of $\uparrow$ cannot cancel out.

$\Rightarrow \deg_\theta b = 0 \Rightarrow b \in F \Rightarrow V_0(\theta) \in F[\theta]$.

$$(*)' \quad P = p_\ell \theta^\ell + \cdots + p_0 \ne \underset{\text{by Th 12.2 } \in F[\theta]}{V_0'(\theta)} + \sum c_i \frac{V_i'(\theta)}{V_i(\theta)} \leftarrow \deg n_i - 1 \text{ by Th 12.2}$$

$\uparrow$ monic irreducible $\deg n_i$

If $\deg_\theta V_i > 0$ then $V_i(\theta)^{-1}$ cannot cancel out $\Rightarrow v_i \in F$.

$$\Rightarrow \int P = \underset{\in F[\theta]}{V_0(\theta)} + \sum c_i \log \underset{\in F}{V_i}$$

$\int P + \frac{1}{x-1}$ $\quad \log(x-1)$

$$\Rightarrow P = p_\ell \theta^\ell + \cdots + p_0 = \underset{\in F[\theta]}{V_0'(\theta)} + \underset{\in F}{\sum L'}$$

Th 12.2 $\Rightarrow \deg_\theta V_0 \le \ell + 1$ hence

$$\int P = \int p_\ell \theta^\ell + \cdots + p_0 = \underset{\in K}{q_{\ell+1}} \theta^{\ell+1} + \underset{\in F}{q_\ell} \theta^\ell + \cdots + \underset{\in F}{q_0} + \sum \underset{\in \bar{K}}{c_i} \log \underset{\in F}{v_i}$$

$$\int p_\ell \theta^\ell + \cdots + p_0 = \underset{\hat{K}}{q_{\ell+1}} \theta^{\ell+1} + \underset{\hat{F}}{q_\ell}\theta^\ell + \cdots + \underset{\hat{F}}{q_1}\theta + \underset{\hat{F}}{q_0} + \Sigma L$$

Differentiating and equating coefficients in θ^i yields

in θ^ℓ	$p_\ell = (\ell+1)\, q_{\ell+1}\theta' + q_\ell'$	(ℓ)
in $\theta^{\ell-1}$	$p_{\ell-1} = \ell q_\ell \theta' + q_{\ell-1}'$	$(\ell-1)$
$\vdots$		
in θ^1	$p_1 = 2q_2\theta' + q_1'$	(1)
in θ^0	$p_0 = q_1\theta' + q_0' + \Sigma L'$	(0)

Integrating both sides of (ℓ) yields

$$\underset{\hat{F}}{\int p_\ell} = \underset{\hat{K}}{\underline{(\ell+1) q_{\ell+1}}}\,\theta + \underset{\hat{F}}{q_\ell}$$

[Liouville $\Rightarrow$ if $\int P$ is elementary
Then $\int p_\ell$ is of this form]

Compute $\int p_\ell$ recursively in F.
If $\int p_\ell$ is not elementary then $\int P$ is not elementary.
If $\int p_\ell$ is elementary and $\int p_\ell = c\log v + \cdots$ and $\log v \notin F(\theta)$
then $\int p_\ell$ is not elementary.
Otherwise $\underset{\text{Liouville}}{\Rightarrow}$ $\int p_\ell = \underset{\hat{F}}{v_\ell} + \underset{\hat{K}}{b_\ell} + c_\ell\theta = (\ell+1)q_{\ell+1}\theta + q_\ell$

Solving for $q_{\ell+1}, q_\ell \Rightarrow q_{\ell+1} = c_\ell/\ell+1$ and $q_\ell = v_\ell + b_\ell$
Substitute $q_\ell = \underset{\hat{F}}{v_\ell} + \underset{\hat{K}}{b_\ell}$ into $(\ell-1)$ yields

$$p_{\ell-1} = \ell(v_\ell + b_\ell)\theta' + q_{\ell-1}'$$
$$\Rightarrow p_{\ell-1} - \ell v_\ell \theta' = \ell b_\ell \theta' + q_{\ell-1}'$$
$$\Rightarrow \int \underline{p_{\ell-1} - \ell v_\ell\theta'} = \ell b_\ell \theta + q_{\ell-1} \quad \text{Repeat!}$$

Example 1. $\int x \log x \, dx = \int x\theta \, dx$ see $\underline{F}(\theta) = \underline{\mathbb{Q}(x)}(\log x)$.

L.T. $\Rightarrow$ $\int x\theta = \underset{\hat{\mathbb{Q}}}{q_2}\theta^2 + \underset{\hat{\mathbb{Q}(x)}}{q_1}\theta + \underset{\hat{\mathbb{Q}(x)}}{q_0} + \Sigma \underset{\hat{\mathbb{C}}}{c_i} \log \underset{\hat{\mathbb{C}[x]}}{v_i}$.

$$\overset{\cap}{\mathbb{Q}} \qquad \overset{\cap}{\mathbb{Q}(x)} \qquad \overset{\cap}{\mathbb{Q}(x)} \qquad \overset{\cap}{\mathbb{C}} \qquad \overset{\cap}{\mathbb{C}[x]}.$$

$$[\theta^1] \quad (1) \qquad x = 2q_2\theta' + q_1'$$

$$[\theta^0] \quad (2) \qquad 0 = q_1\theta' + q_0' + \Sigma L'$$

$$\int(1) \qquad \underline{\tfrac{1}{2}x^2 + b_1} = 2\underset{\overset{\cap}{\mathbb{Q}}}{q_2}\underline{\theta} + \underset{\overset{\cap}{\mathbb{Q}(x)}}{q_1} \Rightarrow \begin{array}{l} q_2 = 0 \\ q_1 = \tfrac{1}{2}x^2 + b_1. \end{array}$$

$$(0) \Rightarrow \qquad 0 = \left(\tfrac{1}{2}x^2 + b_1\right)\cdot\theta' + q_0' + \Sigma L'$$

$$\Longrightarrow \quad -\tfrac{1}{2}x = b_1\theta' + q_0' + \Sigma L'$$

$$\int(0) \Rightarrow \quad -\tfrac{1}{4}x^2 + b_0 = b_1\theta + q_0 + \Sigma L. \Rightarrow \begin{array}{l} b_1 = 0 \\ q_0 = -\tfrac{1}{4}x^2 + b_0 \\ \Sigma L = 0. \end{array}$$

$$\int x \log x \, dx = \overset{q_1\theta}{\left(\tfrac{1}{2}x^2 + 0\right)\log x} - \tfrac{1}{4}x^2 + b_0$$

$\int$ by parts.

Example 2.

$$\int \frac{1}{x}\log x + \frac{1}{x-1}\,dx = \int \frac{1}{x}\theta' + \frac{1}{x-1}\,dx \qquad F(\theta) = \mathbb{Q}(x)(\log x), \quad \frac{1}{x}\theta' + \frac{1}{x-1} \in F[\theta].$$

$$\int \frac{1}{x}\theta' + \frac{1}{x-1}\,dx = \underset{\overset{\cap}{\mathbb{Q}}}{q_2}\theta^2 + q_1(x)\theta + q_0(x) + \Sigma L$$

$$[\theta^1] \Rightarrow \quad \frac{1}{x} = 2q_2\theta' + q_1' \qquad (1)$$

$$[\theta^0] \Rightarrow \quad \frac{1}{x-1} = q_1\theta' + q_0' + \Sigma L' \qquad (0)$$

$$\int(1) \Rightarrow \log x + b_1 = 2q_2\theta + q_1 \Longrightarrow q_2 = \tfrac{1}{2} \quad q_1 = b_1$$

$$\frac{1}{x-1} = b_1\theta' + q_0' + \Sigma L'$$

$$\log(x-1) + b_0 = b_1\theta + q_0 + \Sigma L \Rightarrow b_1 = 0,\ q_0 = b_0,\ \Sigma L = \log(x-1).$$

$$\int \frac{1}{x}\log x\,dx = \frac{1}{2}\log^2 x + 0\cdot\log x + b_0 + \log(x-1).$$

Example 3. $\int (e^{x^2}\log x - \frac{1}{x}e^{x^2})dx$

$F(\theta) = \mathbb{Q}(x)(\theta_1 = e^{x^2})(\log x)$ or $F(\theta) = \mathbb{Q}(x)(\theta_1 = \log x)(e^{x^2})$

Exp. subcase 12.7.

$$\int e^{x^2}\theta' - \frac{1}{x}e^{x^2}dx. = q_2\theta^2 + q_1\theta + q_0 + \Sigma L.$$

$[\theta']$ (1): $e^{x^2} = 2q_2\theta' + q_1'.$

$\int(1) \Rightarrow \int e^{x^2}dx = 2q_2\theta + q_1 \Rightarrow \int f(x)dx$ is not elementary.

not elementary

Example. $\int e^x\log x + \log x + \frac{e^x}{x}\,dx$

$F(\theta) = \mathbb{Q}(x)(\theta_1 = e^x)(\log x)$. $F(\theta) = \mathbb{Q}(x)(\theta_1 = \log x)(\underline{e^x})$.

F

$$\int (e^x+1)\theta + \frac{e^x}{x} = q_2\theta^2 + q_1(x)\theta + q_0(x) + \Sigma L.$$

$'$

$[\theta']$ $e^x + 1 = 2q_2\theta' + q_1'$ (1)

$[\theta^0]$ $\frac{e^x}{x} = q_1\theta' + q_0' + \Sigma L'$ (0)

$\int(1) \Rightarrow e^x + x + b_1 = 2q_2\theta + q_1(x) \Rightarrow q_2 = 0,\ q_1 = e^x + x + b_1,\ \Sigma L = 0.$

$$\Rightarrow \frac{e^x}{x} = (e^x + x + b_1)\theta' + q_0' + \Sigma L'$$

$$\Rightarrow \cancel{\frac{e^x}{x}} - \cancel{\frac{e^x}{x}} - 1 = b_1\theta' + q_0' + \Sigma L' \qquad (0)$$

$$\int(0) \quad \Rightarrow \quad \underline{-x + b_0} = b_1\theta + q_0 + \Sigma L \Rightarrow \begin{array}{l} b_1 = 0 \\ q_0 = -x + b_0 \\ \Sigma L = 0 \end{array}$$

$$\int (e^x+1)\log x + \frac{e^x}{x}\,dx = (e^x + x + 0)\cdot\log x \quad -x + b_0 + 0.$$

Example 5. $\displaystyle\int \log \underbrace{\log x}_{u}\,dx \qquad F(\theta) = Q(x)(\theta_1 = \log x)(\log\log x).$

$$\int 1\cdot\theta'\,dx = \underset{\substack{\Uparrow \\ Q}}{q_2}\theta^2 + \underset{\substack{\Uparrow \\ F}}{q_1}\theta + \underset{\substack{\Uparrow \\ F}}{q_0} + \Sigma L$$

$$\Big\downarrow \text{diff.}$$

$$[\theta^1] \qquad 1 = 2q_2\theta' + q_1 \qquad (1)$$

$$[\theta^0] \qquad 0 = q_1\theta' + q_0' + \Sigma L' \qquad (0)$$

$$\int(1) \qquad x + b_1 = 2q_2\theta + q_1 \quad \Rightarrow q_2 = 0,\ q_1 = x + b_1$$

$$\underset{(0)}{\Rightarrow} \qquad 0 = (x+b_1)\theta' + q_0' + \Sigma L' \qquad\qquad \begin{array}{l} \theta = \log\log x \\ \theta' = \frac{1}{\log x}\cdot\frac{1}{x}. \end{array}$$

$$-x\cdot\frac{1}{\log x}\cdot\frac{1}{x} = b_1\theta' + q_0' + \Sigma L'$$

$$\underset{(0)}{\Rightarrow} \qquad -\frac{1}{\log x} = b_1\theta' + q_0' + \Sigma L'$$

$$\int(0) \quad \Rightarrow \quad -\int\frac{1}{\log x} = b_1\theta + q_0 + \Sigma L.$$

NOT ELEMENTARY. STOP.