Lec25A Risch: exponential subcase: Michael Monagan

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Risch Algorithm: the exponential subcase 12.7.

Let $f(x) \in F(\theta)$ where $F = K(x)(\theta_1, \dots, \theta_n)$ and $\leftarrow$ exp, log, alg.

$\theta = e^w$ where $w \in F$, $w' \neq 0$, θ is NOT algebraic over F.

WLOG $\int \underset{F(\theta)}{\underset{\cap}{f(x)}} dx = \int \frac{a(\theta)}{b(\theta)}$ where $a, b, \in F[\theta]$, $lc_\theta(b) = 1$, $\gcd(a,b) = 1$.

Let $\underline{a = bP + R}$ where $R = 0$ or $\deg_\theta R < \deg_\theta b$

$\Rightarrow \int \frac{a}{b} = \int P(\theta) + \int \frac{R(\theta)}{b(\theta)}$

$$\int \frac{dx}{e^{2x}(e^x + x)} \underset{F(\theta) = Q(x)(e^x)}{=} \int \frac{dx}{\theta^2(\theta + x)} \overset{??}{=} A + \int \frac{C}{\theta(\theta + x)} dx$$

Problem: Let $b(\theta) = \prod_{i=1}^{k} q_i^i$ be the square-free factorization of $b \in \underline{F[\theta]}$

Does $\gcd(q_k, \frac{dq_k}{d\theta}) = 1 \overset{?}{\Rightarrow} \gcd(q_k, \frac{dq_k}{dx}) = 1$

If not can't solve $\sigma q_k' T + T q_k = R$ in $F[\theta]$.

Example. $q_2 = \underline{\theta} = e^{w \in F}$ $\theta' = \underset{F}{\underset{\in}{w'}} \theta \in F[\theta]$.

$\gcd(q_2, \frac{dq}{d\theta}) = \gcd(\theta, 1) = 1.$

$\gcd(q_2, \frac{dq}{dx}) = \gcd(\theta, w'\theta) = \theta.$

Theorem 12.8 Let $b \in F[\theta]$ as above. If $\theta \nmid b(\theta)$ then
$\gcd(b, \frac{db}{d\theta}) = 1 \Rightarrow \gcd(b, \frac{db}{dx}) = 1$ in $F[\theta]$.

Solution. Let $b(\theta) = \theta^l \bar{b}(\theta)$ where $\theta \nmid \bar{b}(\theta)$.

Write $\frac{R(\theta)}{b(\theta)} = \frac{\bar{w}}{\theta^l} + \frac{\bar{R}(\theta)}{\bar{b}(\theta)}$ for some $\bar{w}, \bar{R} \in F[\theta]$.

Solve $\quad R(\theta) = \bar{\omega}\,\underline{b} + \underline{\bar{R}}\,\theta^{\ell}$ for $\bar{\omega}, \bar{R} \in F[\theta]$ with $\deg_\theta \bar{R} < \deg_\theta \bar{b}$.

$$\int f(x) = \int P + \frac{R}{b} = \int \left(P + \frac{\bar{\omega}}{\theta^{\ell}}\right) + \int \frac{\bar{R}(\theta)}{\bar{b}(\theta)} \quad \text{Th 12.8}$$

H.R.

$$= \int \bar{P}(\theta) + \left(\frac{A(\theta)}{B(\theta)} + \int \frac{C(\theta)}{D(\theta)}\right. \text{ s.t. } \theta \nmid D, \ \gcd\left(D, \frac{dD}{d\theta}\right) = 1.$$

"polynomial part" "logarithmic part"

Where $\bar{P}(\theta) = \frac{\bar{\omega}}{\theta^{\ell}} + P$

$= \boxed{p_{-\ell}\theta^{-\ell} + \cdots +} \boxed{p_0 + \cdots p_m\theta^m}$

Claim: If $\int \bar{P}(\theta)$ or $\int \frac{C(\theta)}{D(\theta)}$ is not elementary then $\int f(x)$ is not elementary.

Example.

$$\int \frac{1}{e^{2x}+e^x} = \int \frac{1}{\theta^2+\theta} = \int \frac{1}{\underline{\theta}(\theta+1)}\,dx$$

$F(\theta) = \underline{Q(x)}(e^x)$

$$\frac{1}{\theta(\theta+1)} = \frac{\bar{\omega}}{\theta} + \frac{\bar{R}}{\theta+1} \Rightarrow 1 = \bar{\omega}(\theta+1) + \bar{R}\cdot\theta$$

Solve $\quad \underset{\sigma=-1}{\bar{R}}\,\theta + \underset{\tau=1}{\bar{\omega}}\,\underline{(\theta+1)} = 1$ with $\deg_\theta \bar{R} < \deg_\theta \theta+1 = 1$ in $F[\theta]$

$$\Rightarrow \frac{1}{\theta(\theta+1)} = \frac{1}{\theta} + \frac{-1}{\theta+1}$$

$$\int \frac{dx}{e^{2x}+e^x} = \int \frac{dx}{e^x} - \int \frac{1}{e^x+1} = \int \frac{1}{\theta} - \int \frac{1}{\theta+1}$$

polynomial part logarithmic part.

Example (Hermite Reduction).

$$\int \frac{e^x+1}{e^{2x}+2xe^x+x^2}\,dx = \int \frac{\theta+1}{\theta^2+2x\theta+x^2} = \int \frac{\theta+1}{(\theta+x)^2}$$

SF factorization.

$F(\theta) = Q(x)(e^x)$

$$\int \frac{\theta+1}{(\theta+x)^2} = \int \frac{P}{Q = q_1 \cdot q_2^2 \cdots q_k^k} \qquad k=2 \quad q_1 = 1 \quad q_2 = \theta + x$$

$$T = Q/q_k^k = 1 \quad q_2' = \theta + 1$$

Solve $\sigma q_k' T + \tau q_k = P$ for $\sigma, \tau \in F[\theta]$ with $\deg_\theta \sigma < \deg_\theta q_k$

$\Longrightarrow \sigma \cdot (\theta+1) \cdot 1 + \tau(\theta+x) = \theta+1$

$\sigma = 1 \qquad \tau = 0$

$$\int \frac{P}{Q} = -\frac{\sigma/(k-1)}{q_k^{k-1}} + \int \frac{\tau + \sigma'/(k-1) \cdot T}{Q/q_k}$$

$$\int \frac{\theta+1}{(\theta+x)^2} = \frac{-1/(2-1)}{(\theta+x)^1} + \int \frac{0+0\cdot}{\theta+x}dx = \frac{-1}{e^x+x}.$$

Maple:

gcdex($q_k' T$, q_k, $\underline{P}$, θ, 'σ', 'τ'); $F[\theta]$

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