7.1 Integration by Parts : e.g. $\int (x+1)\sin x \, dx$

Motivation If $F(x)$ is any antiderivative of $f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$.

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Recall $\frac{d}{dx} f(x)\cdot g(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$\downarrow \int$

$$f(x)\cdot g(x) = \int f'(x) g(x)\,dx + \int f(x)\cdot g'(x)\,dx$$

$$\Rightarrow \boxed{\int f(x)\cdot g'(x) = f(x)\cdot g(x) - \int g(x)\cdot f'(x)\,dx} \qquad \boxed{\int f g' = f\cdot g - \int g\cdot f'}$$

Ex 1. $\int \overset{f}{x}' \overset{g'}{e^x}\,dx = \overset{f}{x}\cdot \overset{g}{e^x} - \int e^x \cdot 1\,dx = x\cdot e^x - e^x + C = (x-1)e^x + C$ ✓

$\Rightarrow g = e^x$ ✓

$\int \overset{g'}{x}' \overset{f}{e^x}\,dx = e^x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot e^x\,dx$ ✗

$\Rightarrow g = \frac{1}{2}x^2$ ✓

$$\boxed{\int f\cdot g' = f\cdot g - \int g\cdot f'}$$

$$\int \underline{\underset{f}{x}\ \underset{g'}{\cos x}}\, dx = x\cdot\sin x - \int \sin x\cdot 1\, dx = x\sin x + \cos x + C$$

$g = \sin x$

check $[x\sin x + \cos x]' = 1\cdot\cancel{\sin x} + \underline{x\cdot\cos x} - \cancel{\sin x}$

$$\int \underset{f}{x}\ \underset{g'}{\sin x}\, dx = x(-\cos x) - \int -\cos x\cdot 1\, dx = -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x + C.$$

$g = -\cos x$

$$\int \underset{g'}{2x}\ \underset{f}{\ln x}\, dx = \ln x\cdot x^2 - \int \overset{g}{x^2}\cdot \overset{f'}{\frac{1}{x}}\, dx = x^2\ln x - \int x\, dx = x^2\ln x - \frac{1}{2}x^2 + C.$$

$\Rightarrow g = x^2$

$$\int \underset{f}{x^2}\ \underset{g'}{e^x}\, dx = x^2\cdot e^x - \int e^x\cdot 2x\, dx = x^2e^x - 2\int \underline{x e^x}\, dx$$

$g = e^x$

$$= x^2e^x - 2(xe^x - e^x) + C$$

$$= (x^2 - 2x + 2)e^x + C$$

$$\int \underset{f}{x^n}\, \underset{g'}{e^x}\, dx = x^n e^x - \int e^x (n \cdot x^{n-1})\, dx \qquad \boxed{\int f \cdot g' = fg - \int g f'}$$

$n \geq 1$

$$= x^n e^x - n \int x^{n-1} e^x\, dx \qquad \text{Reduction Formula.}$$

Notation. $\int \underset{u}{f(x)} \cdot \underbrace{g'(x)\, dx}_{dv} = \underset{u}{f(x)} \cdot \underset{v}{\underline{g(x)}} - \int \underset{v}{g(x)} \cdot \underbrace{f'(x)\, dx}_{du}$ $\qquad \boxed{\int u \cdot dv = u \cdot v - \int v\, du}$

Let $u = f(x)$ $\quad \frac{du}{dx} = f'(x) \Rightarrow du = f'(x)\, dx$

Let $v = g(x)$. $\quad \frac{dv}{dx} = g'(x) \Rightarrow dv = g'(x) \cdot dx$

Ex 6. $\int \overset{g'}{1} \cdot \overset{f}{\ln x}\, dx = x \cdot \ln x - \int \overset{g}{x} \cdot \overset{f'}{\frac{1}{x}}\, dx = x \ln x - x + C$

$$\int \underset{u}{\ln x}\ \underbrace{1 \cdot dx}_{dv} = \overset{u}{\ln x} \cdot \overset{v}{x} - \int \overset{v}{x} \cdot \overset{du}{\frac{1}{x} dx} = x \ln x - x + C.$$

$v = x \quad du = \frac{1}{x} dx$

$\int$ Parts works for $\boxed{\int \text{poly}(x) \cdot \begin{matrix} \sin(ax+b) \\ \cos(ax+b) \\ e^{ax+b} \\ \ln(ax+b) \end{matrix} \, dx}$ $\quad x^3-2x+3$

e.g. $\int (3x^2-2x)\cos(3x)\,dx$

Special Cases.

1. $\int \begin{Bmatrix} \sin x \\ \cos x \\ e^x \end{Bmatrix} \cdot \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx$

$$\boxed{\int f \cdot g' = f \cdot g - \int g \cdot f'}$$

$$\int \underbrace{\cos x}_{g'} \cdot \underbrace{\sin x}_{f} \, dx = \sin x \cdot \sin x - \int \underbrace{\sin x}_{g} \cdot \underbrace{\cos x}_{f'} \, dx$$

$g = \sin x$

$$2\int \cos x \cdot \sin x \, dx = \sin^2 x$$

$$1 \cdot \int \cos x \cdot \sin x \, dx = \boxed{\frac{1}{2}\sin^2 x + C}$$

$$\int \cos x \cdot \sin x \, dx = \int \cancel{\cos x} \cdot u \cdot \frac{du}{\cancel{\cos x}} = \int u \, du = \frac{1}{2}u^2 + c = \boxed{\frac{1}{2}\sin^2 x + c}$$

$u = \sin x \quad \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$

Ex. $\int e^x \sin x \, dx$

②

$$\int u(x)\cdot \ln(v(x))\,dx = g(x)\cdot f(x) - \int g(x)\cdot \frac{v'(x)}{v(x)}\,dx$$

$$\underset{g'}{\uparrow} \qquad \underset{f}{\uparrow}$$

$$[\ln(\underbrace{v(x)}_{u})]' = \frac{1}{v(x)}\cdot v'(x) = \frac{v'(x)}{v(x)}$$

$$\int x \ln(1+x^2)\,dx = \frac{1}{2}x^2 \ln(1+x^2) - \int \overset{g}{\frac{1}{2}x^2}\cdot \overset{f'}{\frac{2x}{1+x^2}}\,dx$$

$$\underset{g'}{\uparrow} \qquad \underset{f}{\uparrow} \quad [\ln(\underbrace{1+x^2}_{u})]' = \frac{1}{1+x^2}\cdot 2x$$

$$\int \frac{x^3}{1+x^2}\,dx = \int \frac{u-1}{u}\,\cancel{x}\,\frac{du}{2\cancel{x}} = \int \frac{u-1}{2u}\,du$$

$$x^3 = x^2 \cdot x$$

$$\boxed{u = 1+x^2} \Rightarrow x^2 = u-1$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$