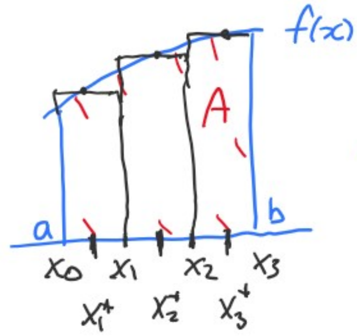
5.3 The Fundamental Theorem of Calculus

January 17, 2024 7:33 AM

Assignment #1 due Monday @ 11pm.



Riemann

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \, f(x_i^*) = \int_a^b f(x)\,dx$$

The definite integral.

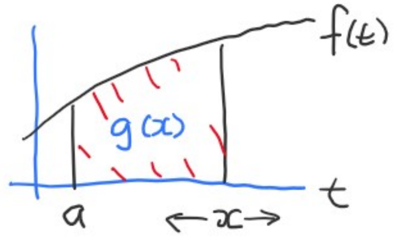
How can we calculate A? Using a limit?

The Fundamental theorem of Calculus (F.T.C.)

Let $f(x)$ be a continuous function on $[a,b]$.

Part I: If $g(x) = \int_a^x \underline{f(t)}\,dt$ then $g'(x) = f(x)$

($g(x)$ is an antiderivative of $f(x)$)



$g(x)$ is an area function

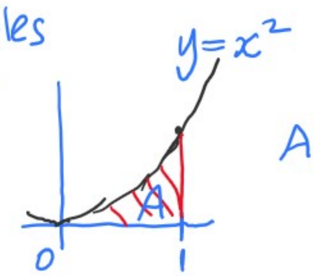
$$g(a) = \int_a^a f(t)\,dt = 0$$

Part II. If $F'(x) = f(x)$ [$F(x)$ is <u>any</u> antiderivative of $f(x)$]

then $$\int_a^b f(x)\,dx = F(\underline{b}) - F(\underline{a}).$$

Examples

①



$$A = \int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \tfrac{1}{3}$$

f(x)

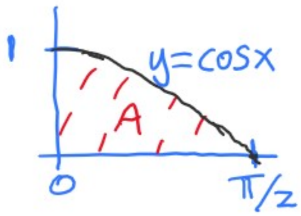
$F(x) = \frac{1}{3}x^3$ $\quad F'(x) = \frac{1}{3}(3x^2) = x^2$

② 

$y=\sqrt{x}$

$$A=\int_0^1 \sqrt{x}\,dx = \overset{F(1)}{\tfrac{2}{3}\cdot 1^{3/2}} - \overset{F(0)}{\tfrac{2}{3}\cdot 0^{3/2}} = \frac{2}{3}$$

$$f(x)=\sqrt{x}=x^{1/2} \qquad F(x)=\frac{2}{3}\cdot x^{3/2}$$

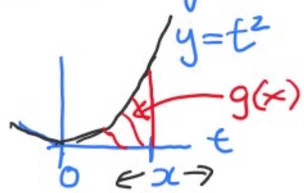
③ 

$$A=\int_0^{\pi/2} \cos x\,dx = \sin\frac{\pi}{2} - \sin 0 = 1-0 = 1$$

$$F(x)=\sin x$$

FTC(1). If $g(x)=\int_a^x f(t)\,dt$ then $g'(x)=f(x)$.

Let $g(x)=\int_0^x t^2\,dt$



$g(0)=\int_0^0 t^2\,dt=0$

FTC(1) $g'(x)=f(x)=\underline{x^2}$

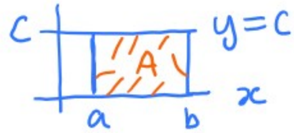
Thm 4.9 $g(x) = \frac{1}{3}x^3 + C$

$g(0) = 0+C = C = 0 \Rightarrow C=0.$

$\Rightarrow g(x) = \frac{1}{3}x^3.$

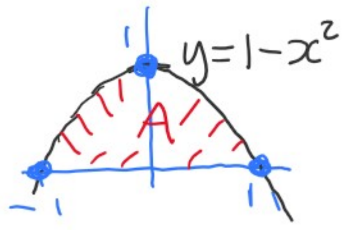
Notation Define $\left[F(x)\right]_a^b = F(b)-F(a)$.

① $$\int_0^1 \underset{\substack{\uparrow\\ f(x)}}{x^2}\,dx = \Big[\underset{\substack{\uparrow\\ F(x)}}{\tfrac{1}{3}x^3}\Big]_0^1 = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}.$$

② 

$A = c\cdot(b-a).$

$$A=\int_a^b \underset{\substack{\uparrow\\ f(x)}}{c}\,dx = \Big[\underset{\substack{\uparrow\\ F(x)}}{cx}\Big]_a^b = c\cdot b - c\cdot a = c(b-a).$$

③ 

$$A=\int_{-1}^{1}\underbrace{(1-x^2)}_{f(x)}dx=\Big[\underbrace{x-\tfrac{1}{3}x^3}_{F(x)}\Big]_{-1}^{1}=\overbrace{\left(1-\tfrac{1}{3}\cdot 1^3\right)}^{F(1)}-\overbrace{\left(-1-\tfrac{1}{3}(-1)^3\right)}^{F(-1)}$$

$$=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$$

④ $A=\int_{-1}^{1}x^2\,dx=\Big[\tfrac{1}{3}x^3\Big]_{-1}^{1}=\tfrac{1}{3}\cdot 1^3-\tfrac{1}{3}\cdot(-1)^3=\frac{2}{3}.$

The FTC Let $f(x)$ be continuous on $[a,b]$.

Part 1. If $g(x)=\int_a^x f(t)\,dt$ then $\boxed{g'(x)=f(x)}$.

Part 2. If $\boxed{F'(x)=f(x)}$ then $\int_a^b f(x)\,dx=F(b)-F(a)$.

✓ (1) ⇒ (2). $F(x)$ and $g(x)$ are antiderivatives of $f(x)$.

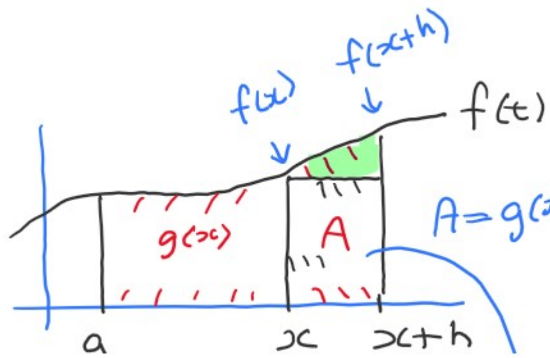
By Theorem 1 of 4.9 $F(x)=g(x)+C$ for some constant C.

$$F(b)-F(a)=(g(b)+C)-(g(a)+C)$$
$$=g(b)-g(a)$$
$$=\int_a^b f(t)\,dt-\underbrace{\int_a^a f(t)\,dt}_{=0}$$
$$=\int_a^b f(x)\,dx.$$ (the variable doesn't matter).

Proof of Part 1.

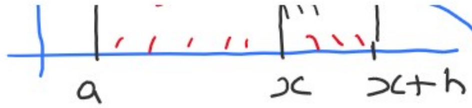
$g(x)=\int_a^x f(t)\,dt.$

To show $g'(x)=f(x)$



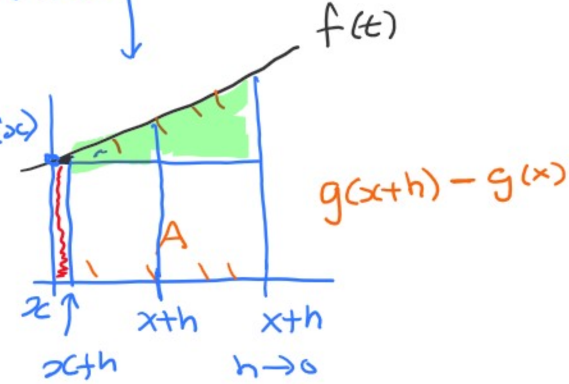
$A=g(x+h)-g(x).$

To show $g'(x)=f(x)$



$A = g(x+h) - g(x) \approx$

$= \underline{h} \cdot f(x).$



$$\Rightarrow \quad \frac{g(x+h)-g(x)}{h} \approx f(x).$$

Take limit as $h \to 0$

$$\Rightarrow \quad \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \overset{?}{=} f(x)$$

$$\Rightarrow \quad \boxed{g'(x) = f(x).}$$

To use the FTC (2) we need an antiderivative of $f(x)$.
How difficult is it to find antiderivatives?

$$\int_1^2 \ln x \, dx =$$