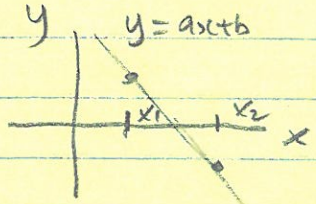
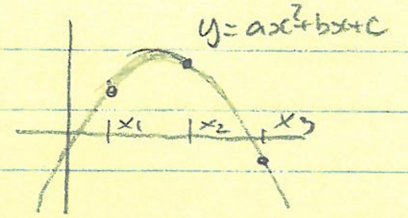
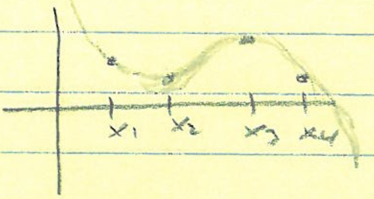
Polynomial Interpolation and The Newton Basis







Theorem. Let $(x_1,y_1),\dots,(x_n,y_n)$ be n points in $\mathbb{R}^2$. If $x_1 \neq x_2 \neq x_3 \neq \dots \neq x_n$ then there exists a unique polynomial $f(x)$ of degree $\leq n-1$ that interpolates the points, i.e., $f(x_1)=y_1,\ f(x_2)=y_2,\dots,\ f(x_n)=y_n$.

$\boxed{\text{Find } f(x)}$

$\{1, x, x^2\}$ standard basis.

$n=3$ $f(x) = C + Bx + Ax^2$

Vandermonde matrix

$$\boxed{\begin{aligned} f(x_1) &= C + Bx_1 + Ax_1^2 = y_1 \\ f(x_2) &= C + Bx_2 + Ax_2^2 = y_2 \\ f(x_3) &= C + Bx_3 + Ax_3^2 = y_3 \end{aligned}} \qquad \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} C \\ B \\ A \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

A linear system in A, B, C $\qquad V_3 \cdot x = y$

If $\det(V_3) \neq 0$ then V_3 is invertible. $\Rightarrow$ $V_3 x = y$ has a unique solution (Th 5 of 2.2) namely $x = V_3^{-1} y$)

Exercise 1. Show that $\det(V_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. Since $x_1 \neq x_2 \neq x_3$ then $\det(V_3) \neq 0$, which proves the Theorem for $n=3$.

Conclusion. To find $f(x)$ just solve $V_3 x = y$.

Exercise 2. Do this for the points $\underline{x} = [1, 0, 2]$, $\underline{y} = [1, -1, 9]$

Newtons Method.

Newton basis $N = \{\overset{b_1}{1}, \overset{b_2}{x - x_1}, \overset{b_3}{(x-x_1)(x-x_2)}, \overset{b_4}{(x-x_1)(x-x_2)(x-3)}, \dots\}$

(n=3) Let $f(x) = c_1 \cdot 1 + c_2(x-x_1) + c_3(x-x_1)(x-x_2)$. [degree 2]

Faster!

$$\begin{cases} f(x_1) = c_1 + c_2 \cdot 0 + c_3 \cdot 0 = y_1 \Rightarrow c_1 = y_1 \\ f(x_2) = c_1 + \boxed{c_2}(x_2 - x_1) + c_3 \cdot 0 = y_2 \Rightarrow c_2 = (y_2 - y_1)/(\underset{\neq 0}{x_2 - x_1}) \\ f(x_3) = c_1 + c_2(x_3 - x_1) + c_3(x_3-x_1)(x_3-x_2) = y_3 \\ c_3 = [y_3 - c_1 - c_2(x_3 - x_1)] / (\underset{\neq 0}{x_3 - x_1})(\underset{\neq 0}{x_3 - x_2}). \end{cases}$$

This proves the existence of f(x) with degree ≤ 2.

The standard basis for $P_2 = \{Ax^2 + Bx + C$ where $A, B, C \in \mathbb{R}\}$ is $\{1, x, x^2\} \Rightarrow \dim P_2 = 3$. Notice b_1, b_2, b_3 are L.I.

Hence N is a basis for P_2. Therefore the solution of $\boxed{f = \underbrace{c_1 b_1 + c_2 b_2 + c_3 b_3}_{\text{basis}}}$ is <u>unique</u> by Th 7 of 4.4.

Suppose we have data $\begin{matrix} (x_1, y_1) & (x_2, y_2) & (x_3, y_3) \\ (1, 1) & (0, -1) & (2, 9) \end{matrix}$

$N = \{1, x-1, (x-1)x\}$

$f = c_1 \cdot 1 + c_2(x-1) + c_3(x-1)x$

$f(1) = c_1 = 1 \Rightarrow c_1 = 1$

$f(0) = c_1 + c_2(-1) = -1 \Rightarrow 1 - c_2 = -1 \Rightarrow c_2 = 2$

$f(2) = 1 + 2(1) + c_3(2) = 9 \Rightarrow 3 + 2c_3 = 9 \Rightarrow c_3 = 3$

What is the answer? $[c_1, c_2, c_3] = [1, 2, 3] \overset{?}{=} [f]_N$ ← co-ordinate vector!

$\Rightarrow \boxed{f(x) = 1 + 2(x-1) + 3(x-1)x}$

$= 1 + 2x - 2 + 3x^2 - 3x = -1 - x + 3x^2$

The vector $[-1, -1, 3] = [f]_B$ $B = \{1, x, x^2\}$ ← standard basis

Exercise 3. Find the change of basis matrix $P_{N \to B}$ such that

$$\underset{P_{N\to B}}{\begin{bmatrix} ■ & ■ & ■ \\ 0 & ■ & ■ \\ 0 & 0 & ■ \end{bmatrix}} \underset{[f]_N}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} = \underset{[f]_B}{\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}}$$

Lagrange's Method $\quad (x_1,y_1),\dots,(x_n,y_n)$

Let $L(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$

Let $L_i(x) = L(x)/(x-x_i) \leftarrow$ degree $n-1$ for $1 \le i \le n$

For $n=3$ use basis $\{L_1(x), L_2(x), L_3(x)\}$ so

$$f(x) = a_1 L_1(x) + a_2 L_2(x) + a_3 L_3(x)$$
$$= a_1(x-x_2)(x-x_3) + a_2(x-x_1)(x-x_3) + a_3(x-x_1)(x-x_2)$$

$$y_1 = f(x_1) = a_1(\overset{\neq 0}{x_1-x_2})(\overset{\neq 0}{x_1-x_3}) + 0 + 0 \Rightarrow a_1 = y_1/(x_1-x_2)(x_1-x_3)$$
$$y_2 = f(x_2) = 0 + a_2(x_2-x_1)(x_2-x_3) + 0 \Rightarrow a_2 = y_2/(x_2-x_1)(x_2-x_3)$$
$$y_3 = f(x_3) = 0 + 0 + a_3(x_3-x_1)(x_3-x_2) \Rightarrow a_3 = y_3/(x_3-x_1)(x_3-x_2).$$

[Solving for a_1, a_2, a_3 is easy.] ✓

To show $\{L_1(x), L_2(x), L_3(x)\}$ is a basis for P_2 we must show these polynomials are linearly independent.

Exercise 4 Show that

$$c_1(x-x_2)(x-x_3)(+ c_2(x-x_1)(x-x_3) + c_3(x-x_1)(x-x_3) = 0$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$.
[Hint this equation must hold for all values of x.]

We have three bases for P_2

$B = \{1, x, x^2\}$ the standard basis,

$N = \{1, x-x_1, (x-x_1)(x-x_2)\}$ the Newton basis and

$L = \{L_1(x), L_2(x), L_3(x)\}$ the Lagrange basis.

Using N and L allows us to find $f(x)$ faster and to prove the Theorem more easlily.