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Model Solutions. Please Return to Michael Monagan

Assignment #1
MATH 439/739.

23 May 2006

MATH 800 Assignment #1

1.1.6. (a) If $f \in \mathbb{C}[x_1,\dots,x_n]$ vanishes at every point of $\mathbb{Z}^n$, then f is the zero polynomial.

Proof: Induction on n.

$n=1$: If $f \in \mathbb{C}[x]$ vanishes at all points of $\mathbb{Z}$ ($f(a)=0\ \forall a \in \mathbb{Z}$) then f has infinitely many roots (since $\mathbb{Z}$ is infinite). By the Fundamental Theorem of Algebra, f must be the zero polynomial in $\mathbb{C}[x]$, since f has more than $\deg(f)$ distinct roots. ✓

$n>1$: Suppose that the statement is true for $n-1$, and let $f \in \mathbb{C}[x_1,\dots,x_n]$ be a polynomial that vanishes at all points of $\mathbb{Z}^n$. Collecting powers of x_n gives

$$f = \sum_{i=0}^{N} g_i(x_1,\dots,x_{n-1})x_n^i$$

where $g_i \in \mathbb{C}[x_1,\dots,x_{n-1}]$.

Given any $(a_1,\dots,a_{n-1}) \in \mathbb{Z}^{n-1}$, we get the polynomial $f(a_1,\dots,a_{n-1},x_n) \in \mathbb{C}[x_n]$. Since f vanishes at all points of $\mathbb{Z}^n$, we have that $f(a_1,\dots,a_{n-1},x_n)$ vanishes $\forall x_n = a \in \mathbb{Z}$. ✓ From the case $n=1$, $f(a_1,\dots,a_{n-1},x_n)$ is the zero polynomial in $\mathbb{C}[x_n]$, so its coefficients $g_i(a_1,\dots,a_{n-1})=0$ for all i. ✓

Because $(a_1,\dots,a_{n-1}) \in \mathbb{Z}^{n-1}$ was arbitrarily chosen, it follows that each $g_i \in \mathbb{C}[x_1,\dots,x_{n-1}]$ vanishes at all points of $\mathbb{Z}^{n-1}$. So by the inductive hypothesis, each g_i is the zero polynomial in $\mathbb{C}[x_1,\dots,x_{n-1}]$. Therefore ✓

$$f = \sum_{i=0}^{N} g_i(x_1,\dots,x_{n-1})x_n^i = \sum_{i=0}^{N} 0\cdot x_n^i = 0.$$

☺ That is, f is the zero polynomial in $\mathbb{C}[x_1,\dots,x_n]$, completing the inductive case.

$\therefore$ by induction, if f vanishes at every point of $\mathbb{Z}^n$ then $f = 0 \in \mathbb{C}[x_1,\dots,x_n]$. □

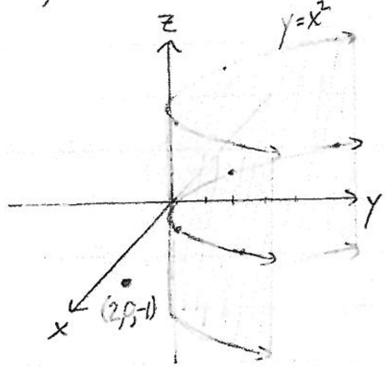
1.2.2. <see Maple>

1.2.4. <see Maple>

1.2.5. Sketch. $V = \mathbb{V}((x-2)(x^2-y),\ y(x^2-y),\ (z+1)(x^2-y))$.

$V = \mathbb{V}(x^2-y) \cup \mathbb{V}(x-2,\ y,\ z+1)$ ✓ (Lemma 2)

So V is the union of the parabolic cylinder defined by $x^2-y=0$ (or $y=x^2$) and the point defined by $x-2=0$, $y=0$, $z+1=0$ (or $x=2$, $y=0$, $z=-1$).



✓

1.2.15. a) Prove that finite unions and intersections of affine varieties are again affine varieties.

Proof: Let $V_1, \dots, V_s \subset k^n$ be affine varieties. We will prove by induction that $\bigcup_{i=1}^{s} V_i$ and $\bigcap_{i=1}^{s} V_i$ are affine varieties.

$s=1$: this is simply the statement V_1 is an affine variety, which is given. ✓

$s=2$: $\bigcup_{i=1}^{2} V_i = V_1 \cup V_2$ and $\bigcap_{i=1}^{2} V_i = V_1 \cap V_2$ are affine varieties by Lemma 2. ✓

$s>2$: Suppose $\bigcup_{i=1}^{s-1} V_i$ and $\bigcap_{i=1}^{s-1} V_i$ are affine varieties.

Then $\bigcup_{i=1}^{s} V_i = \left(\bigcup_{i=1}^{s-1} V_i\right) \cup V_s$ (since $(a_1,\dots,a_n) \in \bigcup_{i=1}^{s} V_i \iff (a_1,\dots,a_n) \in V_s$ or $(a_1,\dots,a_n) \in$ one of $V_1,\dots,V_{s-1}$)

and $\bigcap_{i=1}^{s} V_i = \left(\bigcap_{i=1}^{s-1} V_i\right) \cap V_s$ (since $(a_1,\dots,a_n) \in \bigcap_{i=1}^{s} V_i \iff (a_1,\dots,a_n) \in V_s$ and $(a_1,\dots,a_n) \in$ each of $V_1,\dots,V_{s-1}$).

So by the inductive hypothesis, $\bigcup_{i=1}^{s} V_i$ and $\bigcap_{i=1}^{s} V_i$ are the union and intersection, respectively, of two affine varieties. Then by Lemma 2, $\bigcup_{i=1}^{s} V_i$ and $\bigcap_{i=1}^{s} V_i$ are again affine varieties. ✓

$\therefore$ by induction, $\bigcup_{i=1}^{s} V_i$ and $\bigcap_{i=1}^{s} V_i$ are affine varieties. $\square$ OK

b) An infinite union of affine varieties need not be an affine variety.

Example: Consider the affine varieties $\mathbb{V}(x-a, y-a)$, $a \in \mathbb{R}\setminus\{1\}$. Each of these varieties is a single point $\{(a,a)\}$, $a \neq 1$, and there are an (uncountably) infinite number of them. Taking their union gives

$$\bigcup_{a\in\mathbb{R}\setminus\{1\}} \mathbb{V}(x-a, y-a) = \bigcup_{a\in\mathbb{R}\setminus\{1\}} \{(a,a)\} = \{(a,a) : a \in \mathbb{R}\setminus\{1\}\} \subset \mathbb{R}^2.$$ ✓

This set is the line $y=x$ without the point $(1,1)$. Exercise 1.2.8 showed that this is not an affine variety, as any polynomial that vanishes on this set will also vanish at $(1,1)$. ✓

c) The set-theoretic difference of two affine varieties need not be an affine variety.

Example: $V = \{(a,a) : a \in \mathbb{R}\} \subset \mathbb{R}^2$ is the affine variety $\mathbb{V}(y-x)$. $W = \{(1,1)\}$ is the affine variety $\mathbb{V}(x-1, y-1)$. ✓ But $V - W = \{(a,a) : a \in \mathbb{R}\setminus\{1\}\}$, which again is not an affine variety. ✓

1.3.4. $x = \frac{t}{1+t}, \; y = 1 - \frac{1}{t^2}$

a) Find the equation of the affine variety determined by the parametric equations.

$$x = \frac{t}{1+t} \Rightarrow (1+t)x - t = 0 \Rightarrow (x-1)t + x = 0 \Rightarrow t = \frac{-x}{x-1}$$

Then $\frac{1}{t^2} = \frac{(x-1)^2}{x^2} \Rightarrow y = 1 - \frac{(x-1)^2}{x^2} \Rightarrow (1-y)x^2 = (x-1)^2$

$$\Rightarrow (y-1)x^2 + (x-1)^2 = 0 \Rightarrow yx^2 - 2x + 1 = 0$$

$\therefore$ The variety is $\boxed{\mathbb{V}(yx^2 - 2x + 1)}$ ✓

b) Show that the equations parametrize all points of the variety except $(1,1)$.

From the equation of the variety, $yx^2 - 2x + 1 = 0$, we have $y = \frac{2x-1}{x^2}$.
This is defined for all x except $x = 0$.

Given any $x \neq 0$, we have (from part (a)) that $t = \frac{x}{1-x}$. So unless $x = 1$ (so $y = 1$), there is a value of t such that the parametric equations define the point (x,y) of the variety. ✓

Finally, we can check that the parametric equations for x and y do satisfy the equation of the variety:

$$\left(1 - \frac{1}{t^2}\right)\left(\frac{t}{1+t}\right)^2 - 2\left(\frac{t}{1+t}\right) + 1 = \frac{t^2-1}{t^2} \cdot \frac{t^2}{(t+1)^2} - \frac{2t}{t+1} + 1$$

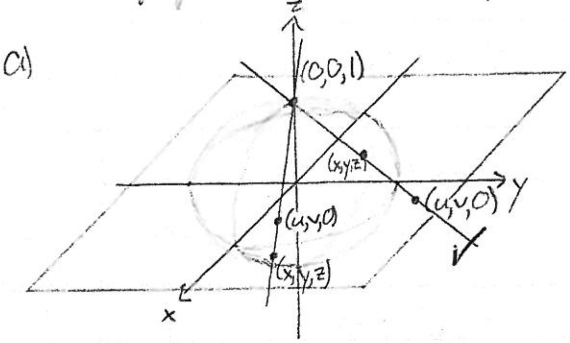
$$= \frac{t-1}{t+1} - \frac{2t}{t+1} + \frac{t+1}{t+1}$$

$$= 0.$$ ✓

⟨see Maple for plots⟩

1.3.6. Parametrizing the sphere $x^2+y^2+z^2=1$

a)



If $(u,v,0)$ lies outside the circle $x^2+y^2=1$ then the line through $(0,0,1)$ and $(u,v,0)$ intersects the upper hemisphere of the sphere in an additional point (x,y,z) between $(0,0,1)$ and $(u,v,0)$. ✓ ✓

If $(u,v,0)$ lies inside the circle $x^2+y^2=1$ then the line through $(0,0,1)$ and $(u,v,0)$ intersects the lower hemisphere of the sphere in an additional point (x,y,z) ~~beyond~~ below $(u,v,0)$. ✓

If $(u,v,0)$ lies on $x^2+y^2=1$ then the line through $(0,0,1)$ and $(u,v,0)$ intersects the sphere at $(u,v,0)$. ✓

Why does this mapping $(u,v)\to(x,y,z)$ parametrize the entire sphere minus $(0,0,1)$? Given any point $(x,y,z)\neq(0,0,1)$ on the sphere we can draw a line descending from $(0,0,1)$ through (x,y,z) that must intersect the xy-plane at some point $(u,v,0)$. ✓

At $(0,0,1)$ the line is tangent to the sphere, lying in the constant $z=1$ plane, so it never intersects the xy-plane. OK

b) The line through $(0,0,1)$ and $(u,v,0)$ is given by

$$(0,0,1)+t\big((u,v,0)-(0,0,1)\big),\quad t\in\mathbb{R} \checkmark$$
$$=(0,0,1)+(tu,tv,-t),\quad t\in\mathbb{R} \checkmark$$
$$=(tu,tv,1-t),\quad t\in\mathbb{R}. \checkmark$$

c) $(tu)^2+(tv)^2+(1-t)^2=1$

$\Rightarrow t^2(u^2+v^2+1)-2t=0$

$\Rightarrow t=0$ or $t=\dfrac{2}{u^2+v^2+1}$ ✓

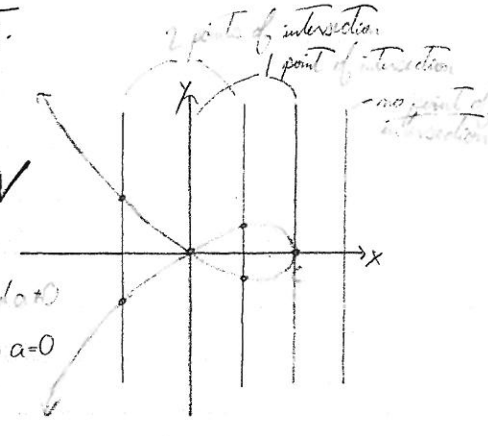
$t=0$ just gives $(x,y,z)=(0,0,1)$, ✓ so we use the other value to find the second point of intersection.

$$\Rightarrow \boxed{\begin{aligned} x&=tu=\frac{2u}{u^2+v^2+1} \checkmark\\ y&=tv=\frac{2v}{u^2+v^2+1} \checkmark\\ z&=1-t=1-\frac{2}{u^2+v^2+1}=\frac{u^2+v^2-1}{u^2+v^2+1} \end{aligned}} \checkmark$$

<see Maple for plots>

1.3.8. $y^2 = cx^2 - x^3$, $c > 0$.

a) A line will meet this curve in 0,1,2, or 3 points.



Case 1: vertical line $x = a$.

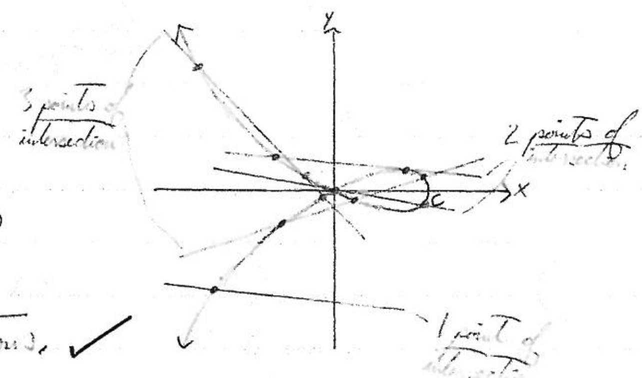
$$y^2 = ca^2 - a^3$$

$$\Rightarrow y = \pm\sqrt{a^2(c-a)}$$

$\Rightarrow$ 2 solutions $y = \pm a\sqrt{c-a}$ if $c > a$ and $a \neq 0$

1 solution $y = 0$ if $c = a$ or $a = 0$

0 (real) solutions if $c < a$.



Case 2: nonvertical line $y = mx + b$.

$$(mx+b)^2 = cx^2 - x^3$$

$$\Rightarrow m^2x^2 + 2mbx + b^2 = cx^2 - x^3$$

$$\Rightarrow x^3 + (m^2 - c)x^2 + 2mbx + b^2 = 0$$

This is a cubic equation in x, so it has 1,2 or 3 real solutions.

$\therefore$ a line will meet the curve in 0,1,2 or 3 points.

b) Show that a nonvertical line through the origin meets the curve in one other point when $m^2 \neq c$.

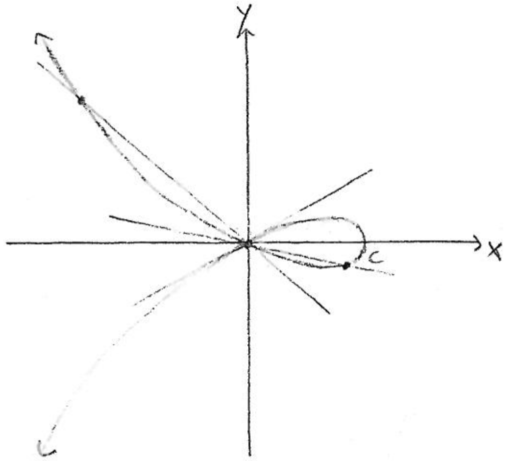
Such a line is given by $y = mx$.
From part (a), this gives

$$x^3 + (m^2 - c)x^2 + 2m(0)x + (0)^2 = 0$$

$$\Rightarrow x^2(x + m^2 - c) = 0$$

$$\Rightarrow x = 0 \text{ or } x = c - m^2$$

$x = 0$ gives the origin, while $x = c - m^2$ ($m^2 \neq c$) gives one other point of intersection.



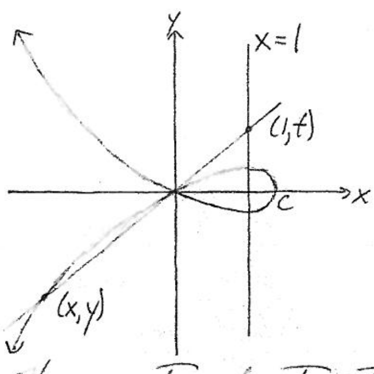
As the line $y = mx$ rotates between the two outer branches of the curve to the left of the origin, the other point of intersection traces out the curve's loop on the right. ✓

When the line is tangent to either branch of the curve where they cross at the origin, the point of intersection is the origin. ✓

+1

As the line rotates further towards vertical, to the right of the origin the line is completely outside the curve's loop. To the left the point of intersection traces out one or the other of the branches (depending on whether $m>0$ or $m<0$) as it extends towards $\pm\infty$. ✓

c)



The line through $(1,t)$ and the origin is simply $y = tx$. So the previous discussion holds with m replaced by t. ✓ As we let $(1,t)$ travel up and down the line $x=1$, the line $y=tx$ rotates through all possible nonvertical lines through the origin. The non-trivial intersection of the curve with $y=tx$ (which we know from part (b) is unique) traces out every point on the curve exactly once as t goes from $-\infty$ to ∞. ✓ ✓

d) From this geometric description, what is the parametrization of x and y in terms of t?

From part (b) we know that intersecting $y=mx$ with the curve gives $x = c-m^2$. Equivalently, intersecting $y=tx$ with the curve gives

$$\boxed{\begin{aligned} x &= c-t^2 \\ y &= tx = t(c-t^2) \end{aligned}}$$ and ✓

<see Maple for plots> ✓

1.3.9. The strophoid: $x = a\cdot\sin(t)$ $\quad -\frac{3\pi}{2} < t < \frac{\pi}{2}$

$y = a\cdot\tan(t)\,(1+\sin(t))$

a) Find the equation in x and y that describes the strophoid.

$$y = a\cdot\frac{\sin(t)}{\cos(t)}(1+\sin(t))$$

$$\Longrightarrow y^2 = \frac{a^2\sin^2(t)\cdot(1+\sin(t))^2}{\cos^2(t)} = \frac{a^2\sin^2(t)\cdot(1+\sin(t))^2}{1-\sin^2(t)}$$

$$= \frac{(a\cdot\sin(t))^2\,(1+\sin(t))^{\cancel{2}\,1}}{(1-\sin(t))\cancel{(1+\sin(t))}}\cdot\frac{a}{a} = \frac{(a\cdot\sin(t))^2(a+a\cdot\sin(t))}{(a-a\sin(t))} ✓$$

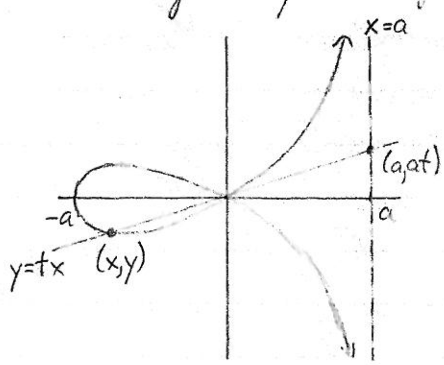
$$= \frac{x^2(a+x)}{(a-x)} ✓$$

$$\Longrightarrow \boxed{(a-x)y^2 = x^2(a+x)} ✓$$

$$\Longrightarrow (a-x)y^2 - x^2(a+x) = 0$$

$\therefore$ The strophoid is $\mathbb{V}((a-x)y^2 - x^2(a+x))$.

b) Find an algebraic parametrization of the strophoid.



We proceed in much the same way as in the parametrization in exercise 8 - start with a vertical line and a point (dependent on t) on this line, and take the line through this point and the origin. The non-trivial intersection of this second line with the strophoid is the point (x,y).

We choose $x=a$, the vertical asymptote of the strophoid, as the vertical line to start with. A point on this line can be described as (a,t) $(-\infty<t<\infty)$, but we will describe it as (a,at) $(-\infty<t<\infty)$ because then the line through this point and the origin is given by

$$y = \frac{at}{a}x = tx. \qquad \text{OK}$$

Now we find the intersection (x,y) of this line and the strophoid.

$$(a-x)y^2 = x^2(a+x)$$
$$\Rightarrow (a-x)(tx)^2 = x^2(a+x)$$
$$\Rightarrow at^2x^2 - t^2x^3 = ax^2 + x^3$$
$$\Rightarrow 0 = (1+t^2)x^3 + a(1-t^2)x^2 = x^2[(1+t^2)x + a(1-t^2)]$$
$$\Rightarrow x = 0 \text{ or } x = \frac{a(t^2-1)}{t^2+1}$$

$x=0$ gives the origin, so the parametrization of the non-trivial intersection (x,y) is

$$\boxed{\begin{aligned} x &= \frac{a(t^2-1)}{t^2+1} \\ y &= tx = \frac{at(t^2-1)}{t^2+1} \end{aligned}}.$$

This is a "rational parametrization." which is algebraic.

<see Maple for plots>

1.3.16. <see Maple>

1.3.17. <see Maple>

1.4.1. ① $x^2+y^2-1=0$, ② $xy-1=0$

a) Eliminate y.

$$\begin{array}{lrl} x\cdot①: & x^3 + xy^2 - x &= 0 \\ -\ y\cdot②: & xy^2 - y &= 0 \\ \hline ③ & x^3 - x + y &= 0 \end{array} \longrightarrow \begin{array}{lrl} x\cdot③: & x^4 - x^2 + xy &= 0 \\ -\ ②: & xy - 1 &= 0 \\ \hline & \boxed{x^4 - x^2 + 1 = 0} \end{array}$$

b) Show how the polynomial in part (a) lies in $\langle x^2+y^2-1, xy-1\rangle$.

Working backwards through the steps in part (a) gives

$$\begin{aligned} x^4 - x^2 + 1 &= x(x^3 - x + y) - (xy-1) \\ &= x[x(x^2+y^2-1) - y(xy-1)] - (xy-1) \\ &= x^2(x^2+y^2-1) - (xy+1)(xy-1) \\ &= h_1\cdot(x^2+y^2-1) + h_2\cdot(xy-1) \in \langle x^2+y^2-1, xy-1\rangle. \end{aligned}$$

1.4.2. Let $I \subset k[x_1,\dots,x_n]$ be an ideal, $f_1,\dots,f_s \in k[x_1,\dots,x_n]$. Prove that the following are equivalent:

(i) $f_1,\dots,f_s \in I$

(ii) $\langle f_1,\dots,f_s\rangle \subset I$.

Proof. (i) $\Rightarrow$ (ii): Let $f \in \langle f_1,\dots,f_s\rangle$. Then $f = \sum_i h_i f_i$, $h_i \in k[x_1,\dots,x_n]$.

$f_1,\dots,f_s \in I \Rightarrow h_1f_1,\dots,h_sf_s \in I$ (by defn. of ideal I)

$\Rightarrow f = \sum_i h_i f_i \in I$. (by defn. of ideal I)

$\therefore \langle f_1,\dots,f_s\rangle \subset I$.

(ii) $\Rightarrow$ (i): $f_i = 0\cdot f_1 + \dots + 0\cdot f_{i-1} + 1\cdot f_i + 0\cdot f_{i+1} + \dots + 0\cdot f_s$

$= \sum_j h_j f_j$ where $h_i = 1$ and $h_j = 0\ \forall j \neq i$

$\in \langle f_1,\dots,f_s\rangle$.

So $\langle f_1,\dots,f_s\rangle \subset I \Rightarrow f_i \in I$.

$\therefore f_1,\dots,f_s \in I$. $\square$

1.4.3. Prove the following equalities of ideals.

a) $\langle x+y, x-y\rangle = \langle x,y\rangle$.

$x+y = 1\cdot x + 1\cdot y \in \langle x,y\rangle$ and $x-y = 1\cdot x + (-1)\cdot y \in \langle x,y\rangle$

$\Rightarrow \langle x+y, x-y\rangle \subset \langle x,y\rangle$.

$x = \frac{1}{2}(x+y) + \frac{1}{2}(x-y) \in \langle x+y, x-y\rangle$ and $y = \frac{1}{2}(x+y) + (-\frac{1}{2})(x-y) \in \langle x+y, x-y\rangle$

$\Rightarrow \langle x,y\rangle \subset \langle x+y, x-y\rangle$.

$\therefore \langle x+y, x-y\rangle = \langle x,y\rangle$.

b) $\langle x+xy, y+xy, x^2, y^2\rangle = \langle x,y\rangle$.

$x+xy = 1\cdot x + x\cdot y \in \langle x,y\rangle$, $y+xy = y\cdot x + 1\cdot y \in \langle x,y\rangle$, $x^2 = x\cdot x + 0\cdot y \in \langle x,y\rangle$ and $y^2 = 0\cdot x + y\cdot y \in \langle x,y\rangle$

$\Rightarrow \langle x+xy, y+xy, x^2, y^2\rangle \subset \langle x,y\rangle$.

$x(y+xy) - y\cdot x^2 = xy \in \langle x+xy, y+xy, x^2, y^2\rangle$

$\Rightarrow x = (x+xy) - xy \in \langle x+xy, y+xy, x^2, y^2\rangle$ and $y = (y+xy) - xy \in \langle x+xy, y+xy, x^2, y^2\rangle$

$\Rightarrow \langle x,y\rangle \subset \langle x+xy, y+xy, x^2, y^2\rangle$.

$\therefore \langle x+xy, y+xy, x^2, y^2\rangle = \langle x,y\rangle$.

c) $\langle 2x^2+3y^2-11,\ x^2-y^2-3 \rangle = \langle x^2-4,\ y^2-1 \rangle$.

$\frac{1}{5}(2x^2+3y^2-11) + \frac{3}{5}(x^2-y^2-3) = \frac{1}{5}(5x^2-20) = x^2-4$ and
$\frac{1}{5}(2x^2+3y^2-11) - \frac{2}{5}(x^2-y^2-3) = \frac{1}{5}(5y^2-5) = y^2-1$
so $x^2-4,\ y^2-1 \in \langle 2x^2+3y^2-11,\ x^2-y^2-3 \rangle$
$\Longrightarrow \langle x^2-4,\ y^2-1 \rangle \subset \langle 2x^2+3y^2-11,\ x^2-y^2-3 \rangle$.

$2(x^2-4)+3(y^2-1) = 2x^2+3y^2-11$ and $(x^2-4)-(y^2-1) = x^2-y^2-3$
so $2x^2+3y^2-11,\ x^2-y^2-3 \in \langle x^2-4,\ y^2-1 \rangle$
$\Longrightarrow \langle 2x^2+3y^2-11,\ x^2-y^2-3 \rangle \subset \langle x^2-4,\ y^2-1 \rangle$.

$\therefore \langle 2x^2+3y^2-11,\ x^2-y^2-3 \rangle = \langle x^2-4,\ y^2-1 \rangle$.

1.4.4. Proposition 4: If $f_1,\dots,f_s$ and $g_1,\dots,g_t$ are bases of the same ideal in $k[x_1,\dots,x_n]$, so that $\langle f_1,\dots,f_s \rangle = \langle g_1,\dots,g_t \rangle$, then $\mathbb{V}(f_1,\dots,f_s) = \mathbb{V}(g_1,\dots,g_t)$.

Proof: Let $(a_1,\dots,a_n) \in \mathbb{V}(f_1,\dots,f_s)$.
Then $f_1,\dots,f_s$ all vanish at $(a_1,\dots,a_n)$.
Now each $g_j \in \langle g_1,\dots,g_t \rangle = \langle f_1,\dots,f_s \rangle$, so
$g_j = \sum_i h_i f_i$ for some $h_i \in k[x_1,\dots,x_n]$.
Then $g_j(a_1,\dots,a_n) = \sum_i h_i \cdot 0 = 0$.
So each g_j also vanishes on $(a_1,\dots,a_n)$.
$\Longrightarrow (a_1,\dots,a_n) \in \mathbb{V}(g_1,\dots,g_t)$.
$\therefore \mathbb{V}(f_1,\dots,f_s) \subset \mathbb{V}(g_1,\dots,g_t)$.

Swapping the roles of $f_1,\dots,f_s$ and $g_1,\dots,g_t$, it follows from the same argument that $\mathbb{V}(g_1,\dots,g_t) \subset \mathbb{V}(f_1,\dots,f_s)$.
$\therefore \mathbb{V}(f_1,\dots,f_s) = \mathbb{V}(g_1,\dots,g_t)$. $\square$

1.4.5. Show that $\mathbb{V}(x+xy,\ y+xy,\ x^2,\ y^2) = \mathbb{V}(x,y)$.

By Exercise 1.4.3(b), $\langle x+xy,\ y+xy,\ x^2,\ y^2 \rangle = \langle x, y \rangle$.
Therefore by Proposition 4, $\mathbb{V}(x+xy,\ y+xy,\ x^2,\ y^2) = \mathbb{V}(x,y)$.

1.4.8. $\mathbb{I}(V)$ and radical ideals.

a) Prove that $\mathbb{I}(V)$ is always a radical ideal.

Proof: Let $V \subset k^n$ be an affine variety and $\mathbb{I}(V)$ be the ideal of V.
Let $f \in k[x_1, \ldots, x_n]$.

OMIT [If $f \in \mathbb{I}(V)$ then clearly $f^m \in \mathbb{I}(V)$ for some positive integer m, namely $m=1$. In fact, $f^m \in \mathbb{I}(V)$ for every $m \in \mathbb{N}$, since $f^m = f^{m-1} \cdot f$ where $f^{m-1} \in k[x_1, \ldots, x_n]$ and $f \in \mathbb{I}(V)$, an ideal.

✓

If $f^m \in \mathbb{I}(V)$ for some positive m, let us assume $m > 1$ (if $m=1$ then $f^m = f$ and we have nothing to prove).
Let $(a_1, \ldots, a_n) \in V$.
Then $(f^m)(a_1, \ldots, a_n) = 0$ (defn. of $\mathbb{I}(V)$)
$\Rightarrow (f(a_1, \ldots, a_n))^m = 0$ ✓
$\Rightarrow f(a_1, \ldots, a_n) = 0$ (since k has no zerodivisors). ✓
Because $(a_1, \ldots, a_n)$ was an arbitrary point of V, it follows that f vanishes at every point of V.
So $f \in \mathbb{I}(V)$. (defn. of $\mathbb{I}(V)$).

✓ $\therefore \mathbb{I}(V)$ is a radical ideal. □

b) Prove that $\langle x^2, y^2 \rangle$ is not a radical ideal.

Proof: Let $f = x$. Then $f^2 = x^2 \in \langle x^2, y^2 \rangle$.
However $f \notin \langle x^2, y^2 \rangle$, because any polynomial $g \in \langle x^2, y^2 \rangle$ has the form $g = h_1 x^2 + h_2 y^2$ $(h_1, h_2 \in k[x,y])$, ✓ so if $g \neq 0$ then every term in g has total degree ≥ 2. But $f = x$ has total degree $= 1$.

OK. $\therefore \langle x^2, y^2 \rangle$ is not a radical ideal. ✓ □

You are using a prop.

1.4.11. $V \subset \mathbb{R}^3$ is parametrized by (t, t^3, t^4).
a) Prove that V is an affine variety.

Proof: $x=t,\ y=t^3,\ z=t^4 \Rightarrow y-x^3=0,\ z-x^4=0$
$\therefore V = \mathbf{V}(y-x^3, z-x^4)$.
So V is an affine variety. $\square$ ✓

b) Determine $\mathbf{I}(V)$.

Note that any monomial $x^\alpha y^\beta z^\gamma$ can be written as
$$x^\alpha y^\beta z^\gamma = x^\alpha (x^3+(y-x^3))^\beta (x^4+(z-x^4))^\gamma$$
$$= x^\alpha (x^{3\beta} + \text{Terms involving } y-x^3)(x^{4\gamma} + \text{Terms involving } z-x^4)$$
$$= h_1(y-x^3) + h_2(z-x^4) + x^{\alpha+3\beta+4\gamma} \quad \text{for some } h_1, h_2 \in \mathbb{R}[x,y,z]$$ OK.

Now any $f \in \mathbb{R}[x,y,z]$ is an $\mathbb{R}$-linear combination of monomials, so f can be written in the form
$$(*) \qquad f = h_1(y-x^3) + h_2(z-x^4) + r$$
where $h_1, h_2 \in \mathbb{R}[x,y,z]$ and $r \in \mathbb{R}[x]$.

Claim: $\mathbf{I}(V) = \langle y-x^3, z-x^4 \rangle$. ✓
Proof: $y-x^3, z-x^4 \in \mathbf{I}(V)$ since they both vanish on V.
$\Rightarrow \langle y-x^3, z-x^4 \rangle \subset \mathbf{I}(V)$. ✓

Let $f \in \mathbf{I}(V)$ and let
$$f = h_1(y-x^3) + h_2(z-x^4) + r$$
be the decomposition given by $(*)$. ✓ $\Rightarrow f(t,t^3,t^4) = 0$
Now $f \in \mathbf{I}(V) \Rightarrow f$ vanishes on V. Since V is parametrized by (t,t^3,t^4) this means that
$$0 \overset{?}{=} f(t,t^3,t^4) \overset{?}{=} h_1(t,t^3,t^4)\cdot\underbrace{(\cancel{t^3-(t)^3})}_{0} + h_2(t,t^3,t^4)\cdot\underbrace{(\cancel{t^4-(t)^4})}_{0} + r(t)$$
$\Rightarrow 0 = 0 + 0 + r(t)$
$\Rightarrow r(t) = 0$.
Since t can be any real numbers, $r(t)$ has infinitely many roots, so $r \in \mathbb{R}[x]$ must be the zero polynomial.
Then $f = h_1(y-x^3) + h_2(z-x^4) \in \langle y-x^3, z-x^4 \rangle$. ✓

Milroy

for $\mathbb{I}(V) \subset \langle y-x^3, z-x^4 \rangle$.

$\therefore \boxed{\mathbb{I}(V) = \langle y-x^3, z-x^4 \rangle}$. $\square$

1.5.8. <see Maple>

1.5.9. <see Maple>

1.5.14. Computing $GCD(f, f')$, $f \in \mathbb{C}[x]$.

a) Prove that if $f = (x-a)^r h$ in $\mathbb{C}[x]$, where $h(a) \neq 0$, then $f' = (x-a)^{r-1} h_1$, where $h_1 \in \mathbb{C}[x]$ does not vanish at a.

Proof: Let $f = (x-a)^r h$ as described.

$$\begin{aligned} f' &= ((x-a)^r)' h + (x-a)^r \cdot h' \\ &= r(x-a)^{r-1} \cdot h + (x-a)^r \cdot h' \\ &= (x-a)^{r-1} [rh + (x-a)h'] \\ &= (x-a)^{r-1} h_1 \quad \text{where } h_1 = rh + (x-a)h'. \end{aligned}$$

$h_1(a) = rh(a) + (\overset{0}{\cancel{a}-a})h'(a) = r \cdot h(a) \neq 0$ because $h(a) \neq 0$. $\square$

b) Prove that if $f = c(x-a_1)^{r_1} \cdots (x-a_l)^{r_l}$ is the factorization of f, where $a_1, \ldots, a_l$ are distinct, then $f' = (x-a_1)^{r_1-1} \cdots (x-a_l)^{r_l-1} \cdot H$, where $H \in \mathbb{C}[x]$ is a polynomial vanishing at none of $a_1, \ldots, a_l$.

Proof: By induction on l.

This is wrong Question assumes $l > 0$.

$l=0$: This is the special case where $f = c$ is a constant. There are no a_i's to worry about. The derivative is $f' = 0 = H$, where there are no a_i's for H to not vanish at.

Base Case.

$l=1$: $f = c(x-a_1)^{r_1}$. Then $f' = cr_1(x-a_1)^{r_1-1}$. Letting $H = cr_1$ (constant) gives $f' = (x-a_1)^{r_1-1} H$, where H does not vanish at a_1, as desired. from part a).

$\ell > 1$: Assume that the statement is true for polynomials with $\ell - 1$ distinct factors. (inductive hypothesis)

$f = c(x-a_1)^{r_1} \cdots (x-a_\ell)^{r_\ell}$.

Let $h = c(x-a_1)^{r_1} \cdots (x-a_{\ell-1})^{r_{\ell-1}}$. Then

$f = h \cdot (x-a_\ell)^{r_\ell} = (x-a_\ell)^{r_\ell} \cdot h$ where $h(a_\ell) \neq 0$ (since a_i's are distinct).

As in the proof of part (a), differentiating gives

$f' = (x-a_\ell)^{r_\ell - 1} \cdot [r_\ell \cdot h + (x-a_\ell) \cdot h']$.

By the inductive hypothesis, $h' = (x-a_1)^{r_1 - 1} \cdots (x-a_{\ell-1})^{r_{\ell-1}-1} \cdot H_1$, ✓

where $H_1 \in \mathbb{C}[x]$ vanishes at none of $a_1, \ldots, a_{\ell-1}$. Then ✓

$$\begin{aligned}
f' &= (x-a_\ell)^{r_\ell - 1} \cdot [r_\ell \cdot c(x-a_1)^{r_1} \cdots (x-a_{\ell-1})^{r_{\ell-1}} + (x-a_\ell)(x-a_1)^{r_1-1} \cdots (x-a_{\ell-1})^{r_{\ell-1}-1} \cdot H_1] \\
&= (x-a_1)^{r_1-1} \cdots (x-a_{\ell-1})^{r_{\ell-1}-1} (x-a_\ell)^{r_\ell - 1} \cdot [r_\ell c (x-a_1)^1 \cdots (x-a_{\ell-1})^1 + (x-a_\ell) \cdot H_1] \\
&= (x-a_1)^{r_1-1} \cdots (x-a_\ell)^{r_\ell-1} \cdot H
\end{aligned}$$

where $H = r_\ell c \cdot \prod_{i=1}^{\ell-1} (x-a_i) + (x-a_\ell) H_1$.

Now if we evaluate H at any of the a_i's, $1 \leq i \leq \ell-1$, we get

$$\begin{aligned}
H(a_i) &= r_\ell c (a_i - a_1) \cdots \overset{0}{(a_i - a_i)} \cdots (a_i - a_{\ell-1}) + (a_i - a_\ell) H_1(a_i) \\
&= 0 + (a_i - a_\ell) \cdot H_1(a_i) \\
&= (a_i - a_\ell) \cdot H_1(a_i)
\end{aligned}$$

✓ $\neq 0$, because $a_1, \ldots, a_\ell$ are distinct, ✓ and H_1 vanishes at none of $a_1, \ldots, a_{\ell-1}$. ✓

Evaluating H at a_ℓ gives

$$\begin{aligned}
H(a_\ell) &= r_\ell c \cdot (a_\ell - a_1) \cdots (a_\ell - a_{\ell-1}) + \overset{0}{(a_\ell - a_\ell)} H_1(a_\ell) \\
&= r_\ell c \cdot \prod_{i=1}^{\ell-1} (a_\ell - a_i)
\end{aligned}$$

$\neq 0$, again because $a_1, \ldots, a_\ell$ are distinct ✓

Therefore $H \in \mathbb{C}[x]$ vanishes at none of $a_1, \ldots, a_\ell$. ✓

This completes the proof that f' has the desired form ✓ and properties in the inductive case. □

c) Prove that $GCD(f,f') = (x-a_1)^{r_1-1} \cdots (x-a_\ell)^{r_\ell-1}$.

Proof: Let $g = (x-a_1)^{r_1-1} \cdots (x-a_\ell)^{r_\ell-1}$.
We have $f = c(x-a_1)^{r_1} \cdots (x-a_\ell)^{r_\ell}$ and $f' = (x-a_1)^{r_1-1} \cdots (x-a_\ell)^{r_\ell-1} \cdot H$
where $H \in \mathbb{C}[x]$ vanishes at none of $a_1, \dots, a_\ell$.
So $g \mid f$ and $g \mid f'$, as both f and f' have at least as many factors of each $(x-\alpha_i)$ as g. ✓

Now let $p \in \mathbb{C}[x]$ be another polynomial that divides both f and f'. Since the only factors of f are of the form $(x-a_i)$, these must be the only factors of p as well. That is, ✓
$p = (x-a_1)^{e_1} \cdots (x-a_\ell)^{e_\ell}$.
assumes k is a field. actually $k[x]$ is a UFD
Note that H does not vanish at $a_i \Rightarrow x - a_i \nmid H$,
so H has no factors of the form $(x-a_i)$.
Thus f' has exactly $r_1 - 1$ factors of $(x-a_1)$, r_2-1 factors of $(x-a_2)$, $\dots$, $r_\ell - 1$ factors of $(x-a_\ell)$.
Since $p \mid f'$, p can't have more factors of any $(x-a_i)$ than f', so $e_i \leq r_i - 1$ for $1 \leq i \leq \ell$.
Then $p \mid g$.

✓

$\therefore g$ satisfies the definition of $GCD(f,f')$. □

1.5.15. Computing f_{red} of $f \in \mathbb{C}[x]$.
a) Prove that $f_{red} = \dfrac{f}{GCD(f,f')}$.

Proof: Let $f = c(x-a_1)^{r_1} \cdots (x-a_\ell)^{r_\ell}$.
In Exercise 12 f_{red}, the square-free part of f, was defined to be $f_{red} = c(x-a_1) \cdots (x-a_\ell)$.
In Exercise 14(c) we showed that $GCD(f,f') = (x-a_1)^{r_1-1} \cdots (x-a_\ell)^{r_\ell-1}$.
Then $\dfrac{f}{GCD(f,f')} = \dfrac{c(x-a_1)^{r_1} \cdots (x-r_\ell)^{r_\ell}}{\cancel{(x-a_1)^{r_1-1}} \cdots \cancel{(x-a_\ell)^{r_\ell-1}}} = c(x-a_1)\cdots(x-a_\ell) = f_{red}$. □

✓

b) ⟨see Maple⟩

1.5.16. Describe an algorithm whose input consists of polynomials $f_1, \dots, f_s \in \mathbb{C}[x]$ and whose output consists of a basis of $\mathbb{I}(\mathbb{V}(f_1, \dots f_s))$.

Algorithm:

1. Compute $g = GCD(f_1, \dots, f_s)$ using the Euclidean algorithm (applied recursively to $f_1, \dots, f_s$, i.e. $g = GCD(f_1, GCD(f_2, \dots GCD(f_{s-1}, f_s)\dots))$, if your implementation of the Euclidean algorithm can't take an arbitrary list as input).
2. Compute g', the derivative of g. (This is a simple exercise in symbolic computation.)
3. Compute the square-free part $g_{red} = \dfrac{g}{GCD(g, g')}$. ✓
4. Return the polynomial g_{red} as the basis.

Justification:

By Proposition 6, §1.5, $\langle g \rangle = \langle f_1, \dots, f_s \rangle$ (note that $\mathbb{C}[x]$ is a principal ideal domain).
Then by Proposition 4, §1.4, $\mathbb{V}(f_1, \dots, f_s) = \mathbb{V}(g)$. So clearly $\mathbb{I}(\mathbb{V}(f_1, \dots, f_s)) = \mathbb{I}(\mathbb{V}(g))$.
By Exercise 1.5.12, $\mathbb{I}(\mathbb{V}(g)) = \langle g_{red} \rangle$ (which, from §1.4, is a radical ideal). So $\mathbb{I}(\mathbb{V}(f_1, \dots, f_s)) = \langle g_{red} \rangle$.
By Exercise 1.5.15, g_{red} can be computed as $g_{red} = \dfrac{g}{GCD(g, g')}$

1.5.17. ⟨see Maple⟩

MATH 800 Assignment 1
(due May 24, 2006, 9:30)

```
> restart;
```

1.2.2

We enter the polynomial defining the variety.

```
> f := y^2-x*(x-1)*(x-2);
```

$$f := y^2 - x\,(x-1)\,(x-2)$$

We can solve $f = 0$ for y exactly when $0 \le y^2$, that is, when $0 \le x\,(x-1)\,(x-2)$.

```
> y^2-f;
  solve( y^2-f >= 0, {x} );
```

$$x\,(x-1)\,(x-2)$$

$$\{0 \le x, x \le 1\}, \{2 \le x\}$$

We can solve for y when x is between 0 and 1 (inclusive), or at least 2.

To solve for y we isolate the y^2 term in $f = 0$ and take the square root of both sides.

```
> isolate( f = 0, y^2 );
  map( sqrt, % );
  solve( simplify(%), {y} );
```

$$y^2 = x\,(x-1)\,(x-2)$$

$$\sqrt{y^2} = \sqrt{x\,(x-1)\,(x-2)}$$

$$\{y = \sqrt{x\,(x-1)\,(x-2)}\}, \{y = -\sqrt{x\,(x-1)\,(x-2)}\}$$

There are 2 y's for each valid x.

We expect there to be symmetry over the x-axis (vertical symmetry), which we verify as follows.

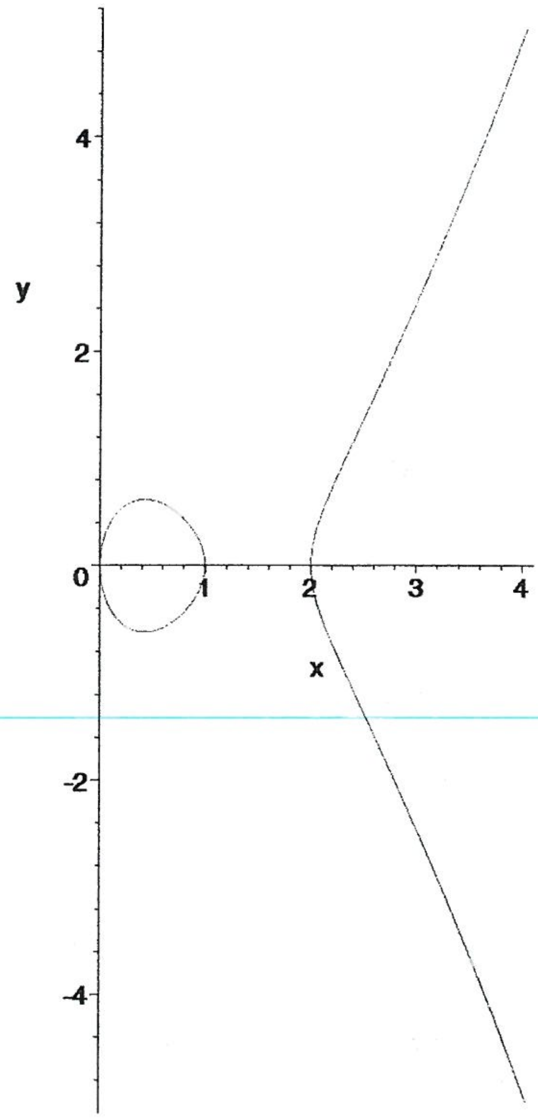
```
> subs( y=-y, f );
```

$$y^2 - x\,(x-1)\,(x-2)$$

Finally we plot the variety V(f).

```
> plots[implicitplot]( f, x=0..5, y=-5..5, grid=[100,100],
  scaling=constrained );
```

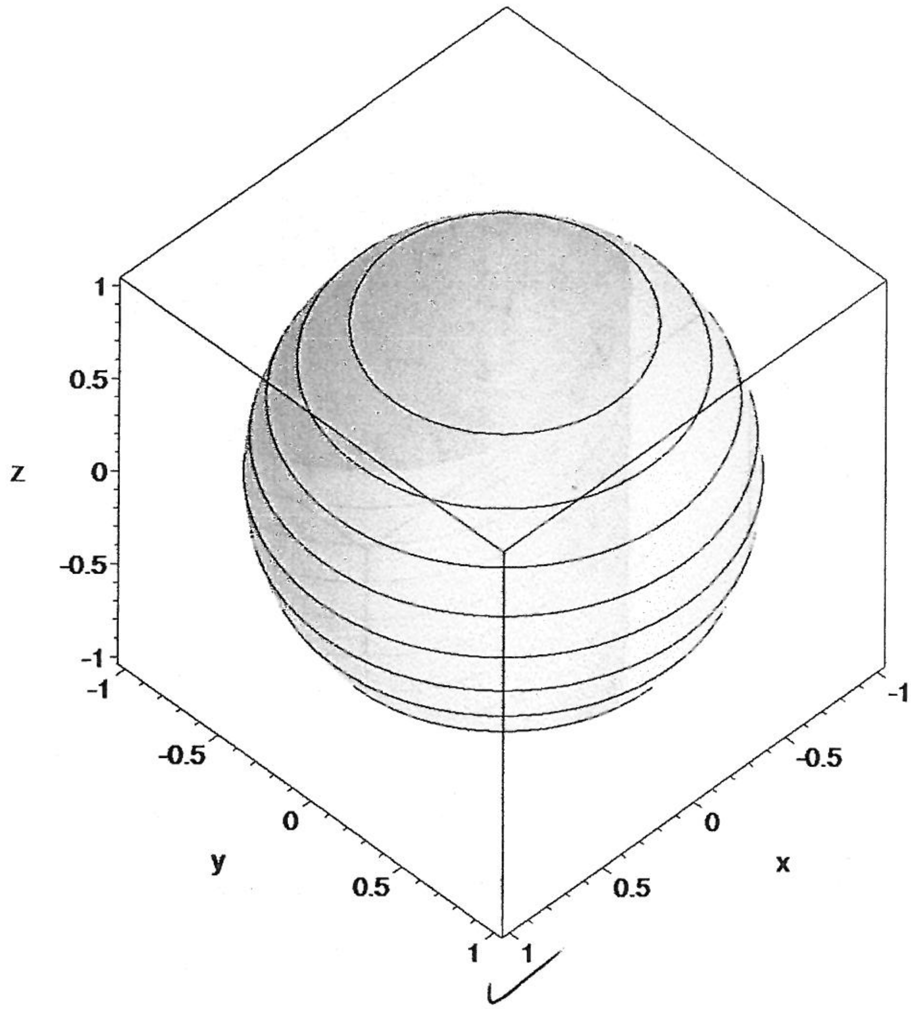
-1..5



1.2.4

a)

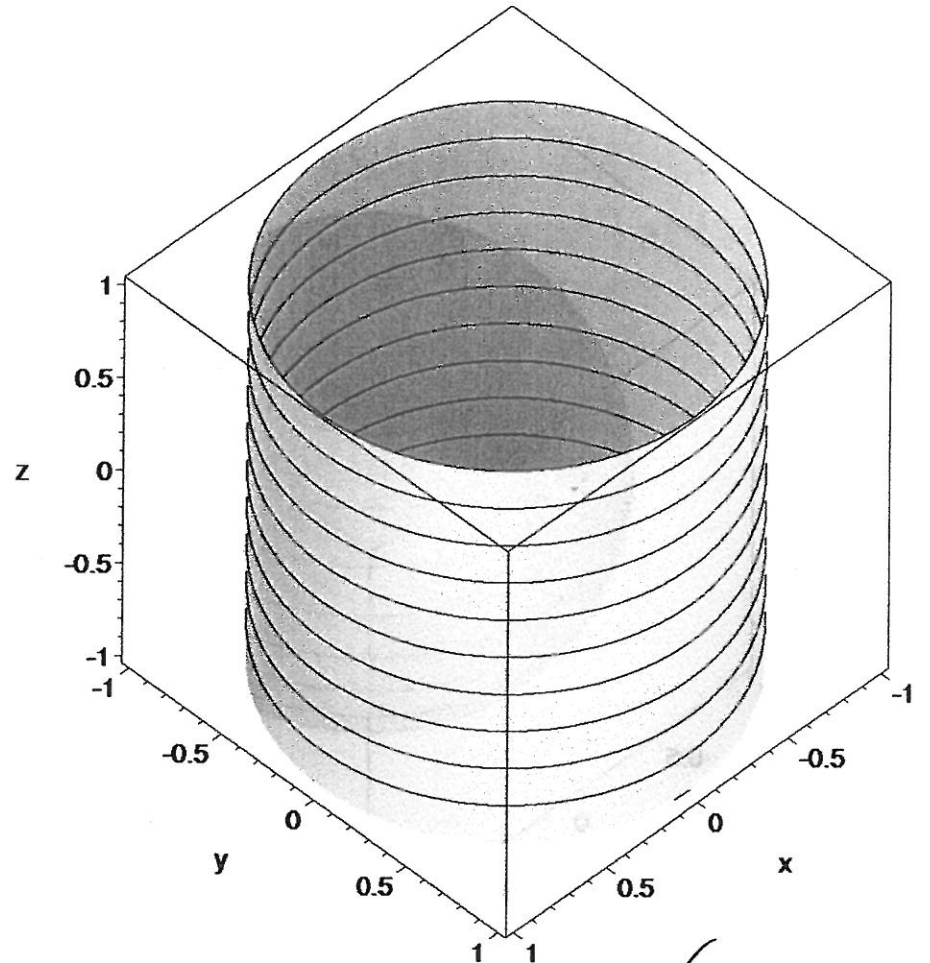
```
> plots[implicitplot3d]( x^2+y^2+z^2-1, x=-1..1, y=-1..1,
  z=-1..1, axes=box, grid=[25,25,25], style=patchcontour,
  scaling=constrained );
```



A sphere of radius 1 centred at the origin.

b)

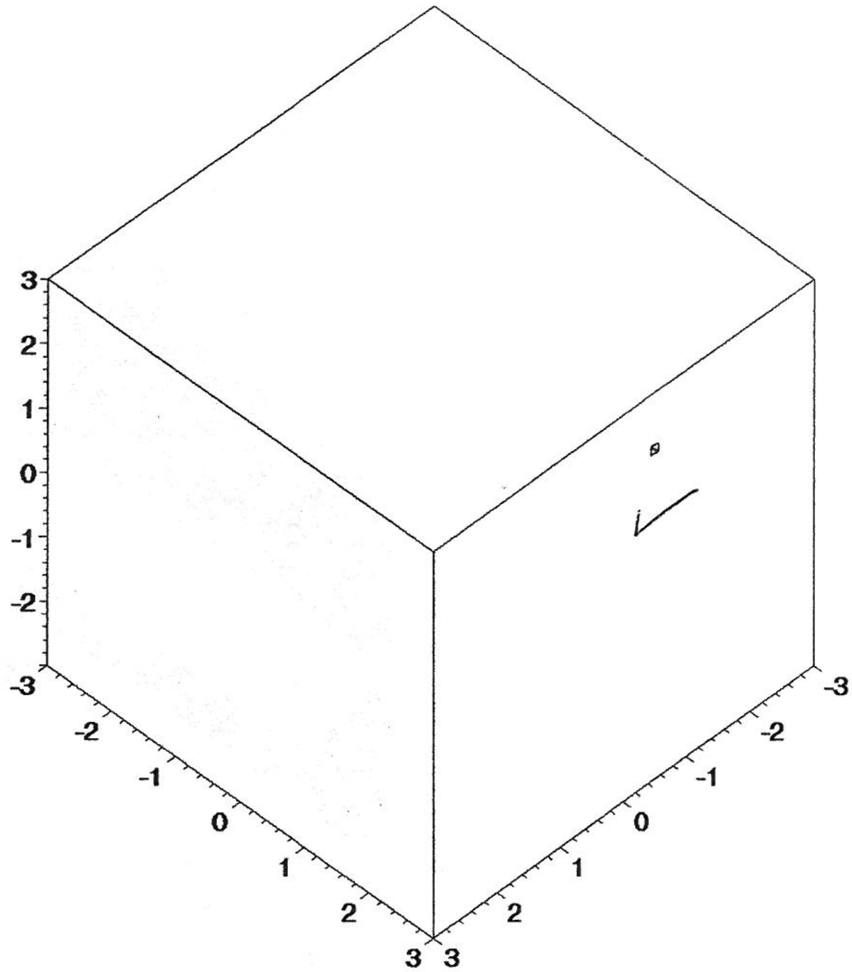
```
> plots[implicitplot3d]( x^2+y^2-1, x=-1..1, y=-1..1, z=-1..1,
  axes=box, grid=[25,25,25], style=patchcontour,
  scaling=constrained );
```



A right circular cylinder of radius 1 centred at the origin and parallel to the *z*-axis.

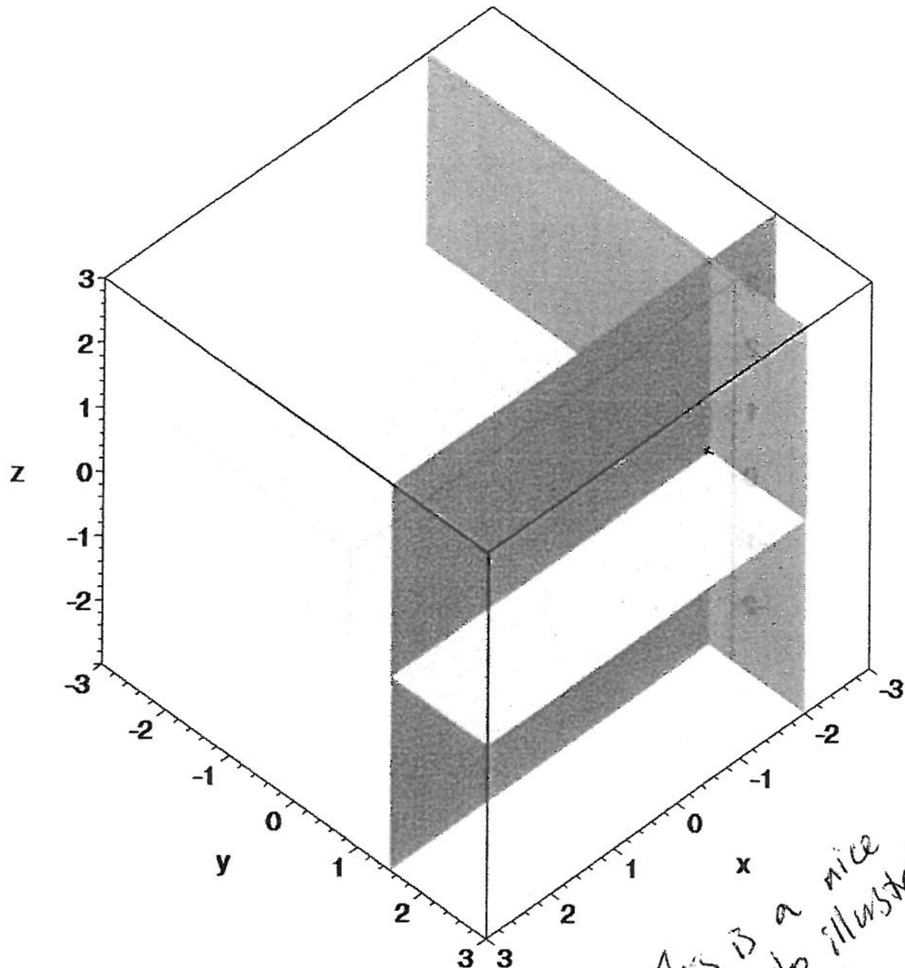
c)

```
> P := plots[pointplot3d]( [-2,1.5,0], color=black,
  symbol=diamond, view=[-3..3,-3..3,-3..3], axes=box,
  scaling=constrained ):
  plots[display]( P );
```



A single point at coordinates [-2, 1.5, 0]. This variety is the intersection of the 3 planes V($x+2$), V($y-1.5$), V(z) shown below.

```
> S := plots[implicitplot3d]( [x+2, y-1.5, z], x=-3..3,
  y=-3..3, z=-3..3, axes=box, color=[cyan,green,yellow],
  grid=[25,25,25], style=patchnogrid, scaling=constrained ):
  plots[display]( P, S );
```



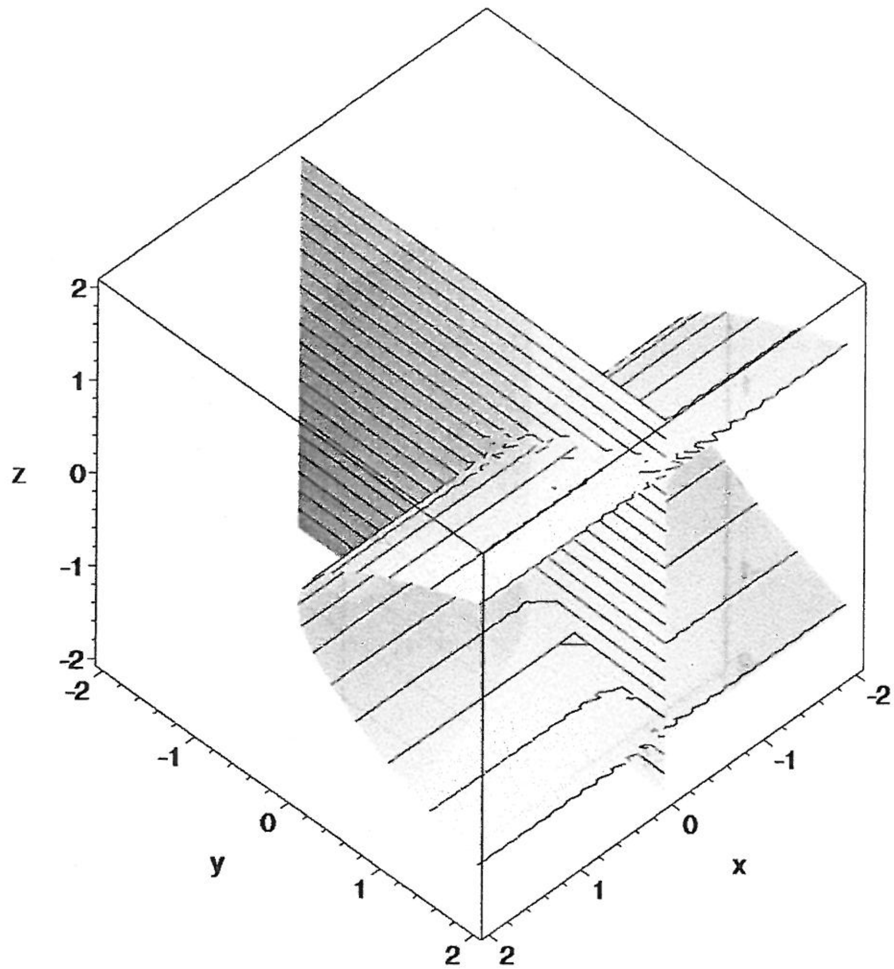
This is a nice way to illustrate a point.

d)

```
> f := x*z^2-x*y;
```

$$f := x\,z^2 - x\,y$$

```
> plots[implicitplot3d]( f, x=-2..2, y=-2..2, z=-2..2,
  axes=box, grid=[25,25,25], style=patchcontour,
  scaling=constrained );
```

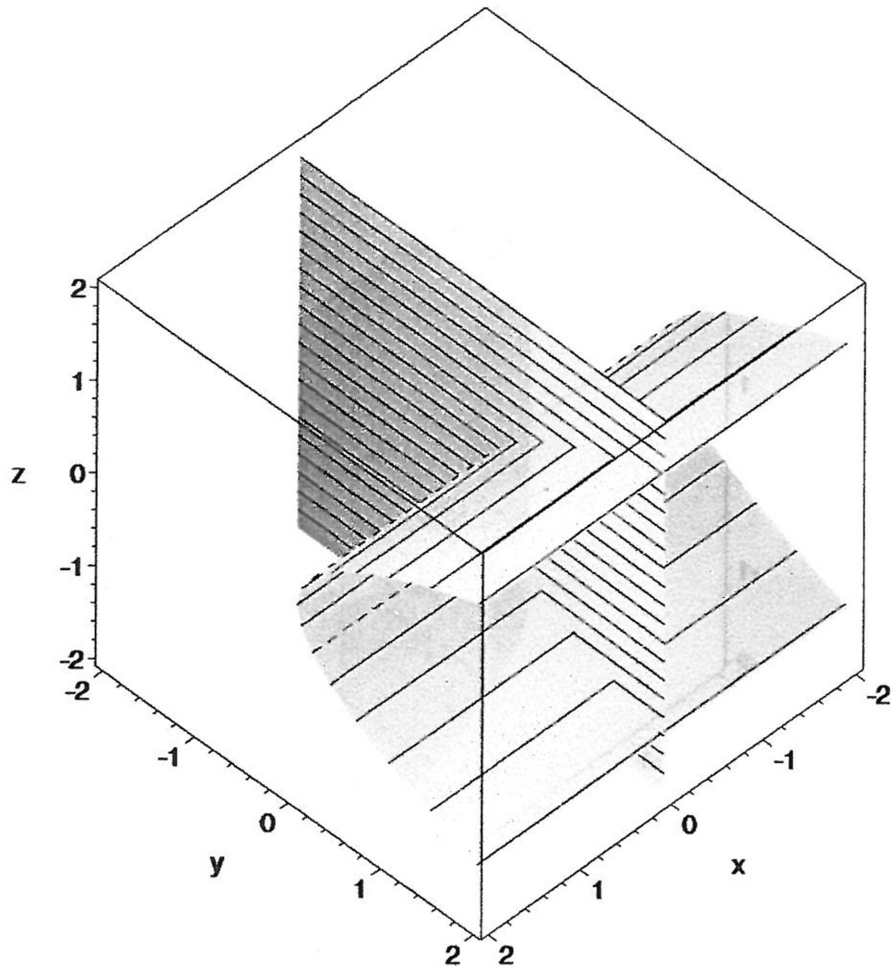


By Lemma 2, this variety is the union of the plane $V(x)$ and the parabolic cylinder $V(z^2 - y)$, as shown below.

```
> factor(f);
```

$$-x\,(-z^2 + y)$$

```
> plots[implicitplot3d]( [op( 2..3, % )], x=-2..2, y=-2..2,
  z=-2..2, axes=box, grid=[25,25,25], style=patchcontour,
  scaling=constrained );
```



✓

e)

```
> (f,g) := x^4-z*x, x^3-y*x;
```

$$f, g := x^4 - z\,x, x^3 - x\,y$$

```
> factor( f );
  factor( g );
  gcd( f, g );
```

$$x\,(x^3 - z)$$

$$x\,(x^2 - y)$$

$$x$$

✓

By factoring f, g we see that $V(f, g)$ is the union of $V(x)$ and $V(x^3 - z, x^2 - y)$ (by Lemma 2). The first variety is a plane, while the second we recognize as the twisted cubic. We verify this by solving for x, y, z.

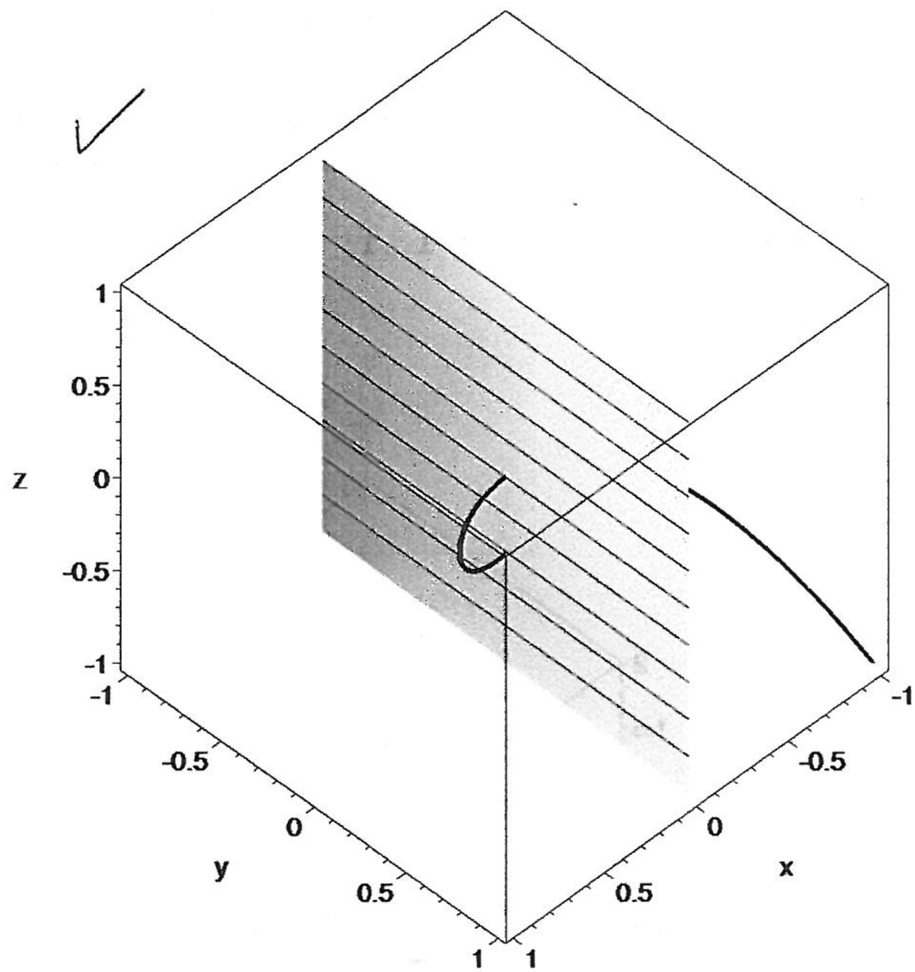
```
> solve( {f/x, g/x}, {x,y,z} );
```

✓

$$\{y = x^2, x = x, z = x^3\}$$

We plot the plane and curve forming the variety.

```
> P := plots[implicitplot3d]( x, x=-1..1, y=-1..1, z=-1..1,
  axes=box, grid=[25,25,25], style=patchcontour,
  scaling=constrained ):
  C := plots[spacecurve]( [x, x^2, x^3], x=-1..1, color=black,
  thickness=5, axes=box, scaling=constrained ):
  plots[display]( P, C );
```



f)

```
> (f,g) := x^2+y^2+z^2-1, x^2+y^2+(z-1)^2-1;
```

$$f, g := x^2 + y^2 + z^2 - 1, x^2 + y^2 + (z-1)^2 - 1$$

The variety V(f, g) is the intersection of the spheres V(f), V(g). We solve for the intersections of these two spheres.

```
> solve( {f = g}, {z} );
  subs( %, {f, g} );
```

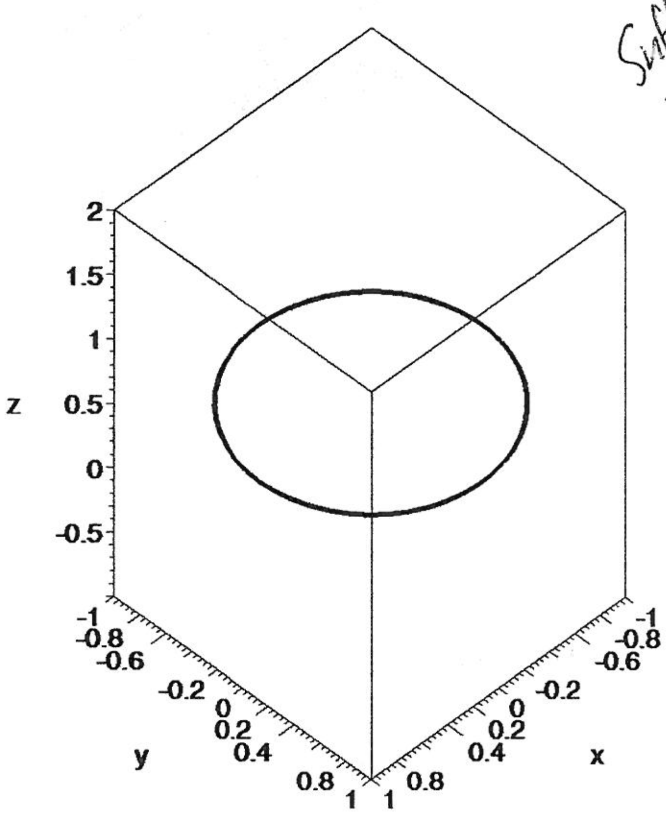
$$\{z = \frac{1}{2}\}$$

$$\{x^2 + y^2 = \frac{3}{4}\}$$

The two spheres are the same except for the z-coordinates of their centres. We thus expect them to intersect at a constant z-value, which we find to be $z = \frac{1}{2}$. The intersection is a circle in this level z-plane, centred at (0, 0) and with radius $\frac{\sqrt{3}}{2}$. We plot this curve using a trigonometric parametrization.

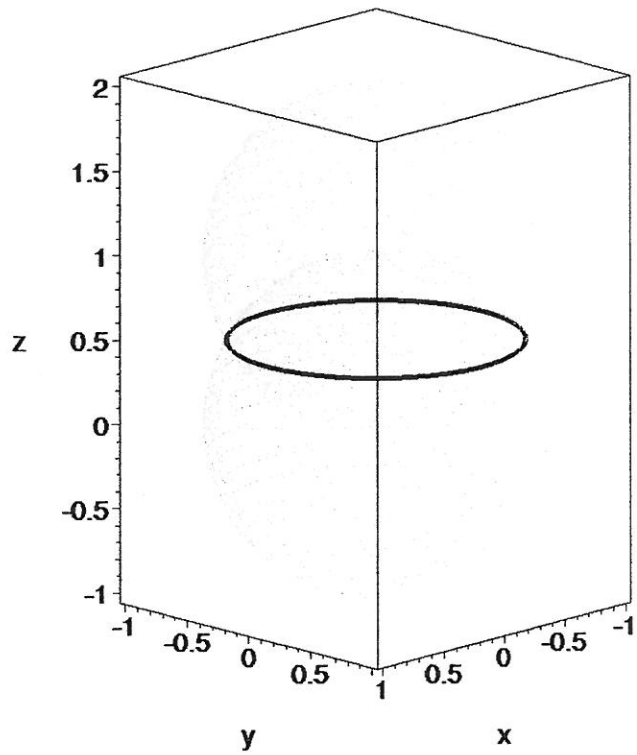
```
> C := plots[spacecurve]( [sqrt(3)/2*cos(t), sqrt(3)/2*sin(t),
  1/2], t=0..2*Pi, color=black, thickness=5, axes=box,
  scaling=constrained ):
  plots[display]( C, view=[-1..1, -1..1, -1..2], labels=[x,y,z]
  );
```

✓ Sufficient to just plot both surfaces in different colors. It's more convincing.



And we plot this variety along with the two spheres that it is the intersection of.

```
> S := plots[implicitplot3d]( [f, g], x=-1..1, y=-1..1,
  z=-1..2, axes=box, grid=[30,30,30], style=point,
  scaling=constrained ):
  plots[display]( C, S, orientation=[45,75] );
```



OK ✓

Scott Cowan
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Missing answer.

```
>
```

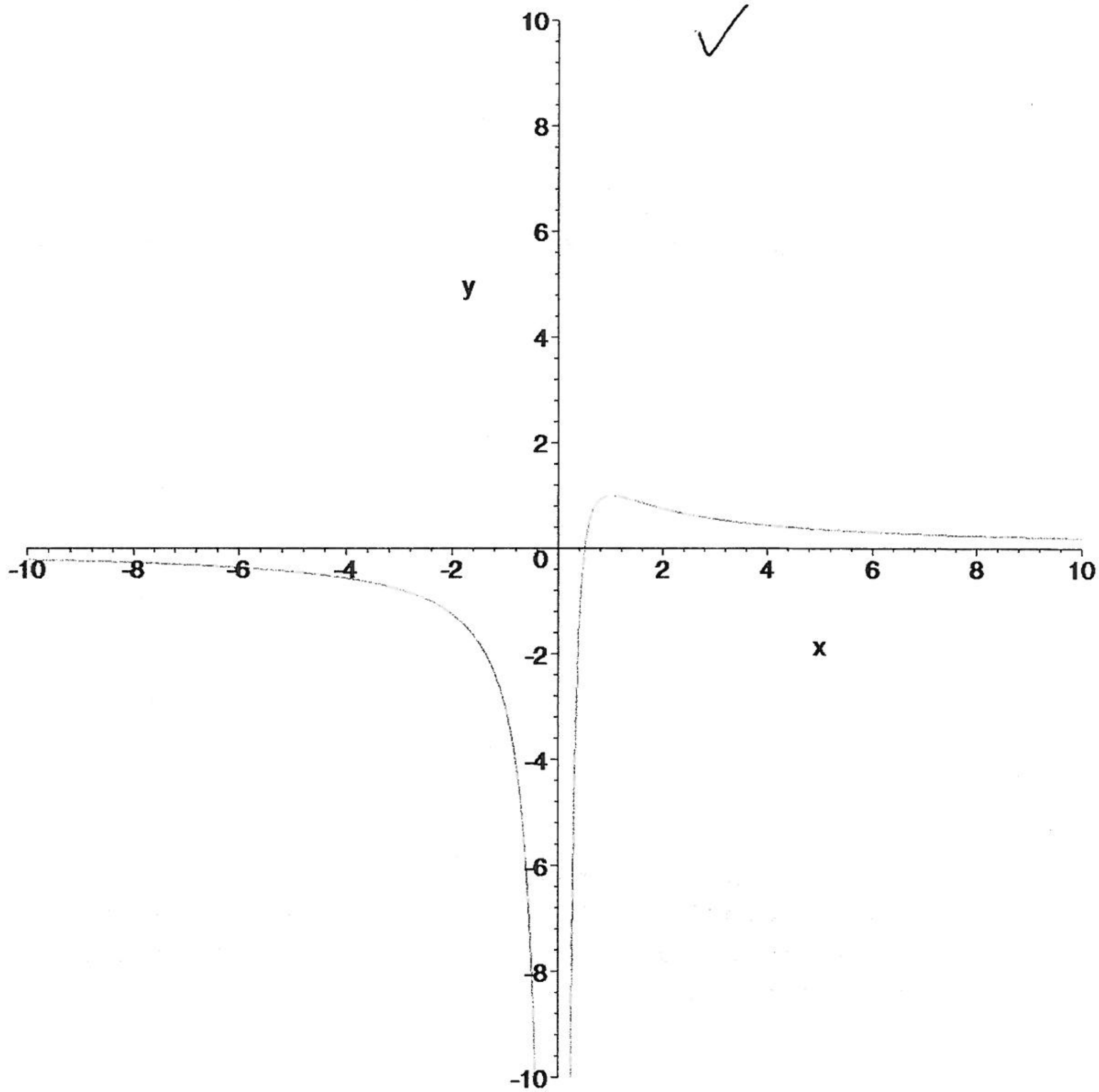
MATH 800 Assignment 1
(due May 24, 2006, 9:30)

```
> restart;
```

1.3.4

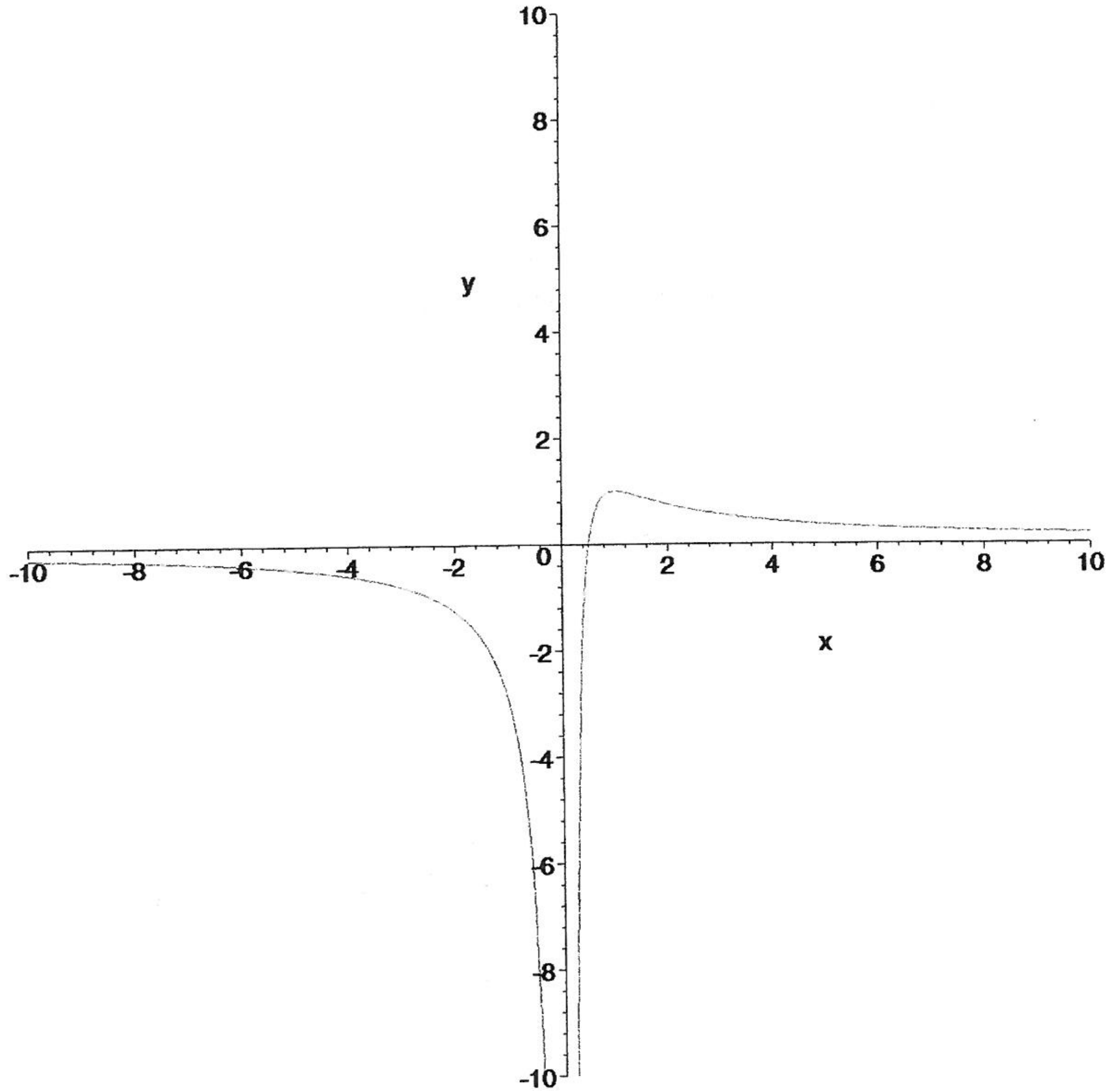
We plot the given parametric representation.

```
> plot( [t/(1+t), 1-1/t^2, t=-20..20], color=blue, discont=true,
  view=[-10..10, -10..10], labels=[x,y], scaling=constrained );
```



We plot our derived polynomial equation that defines the variety.

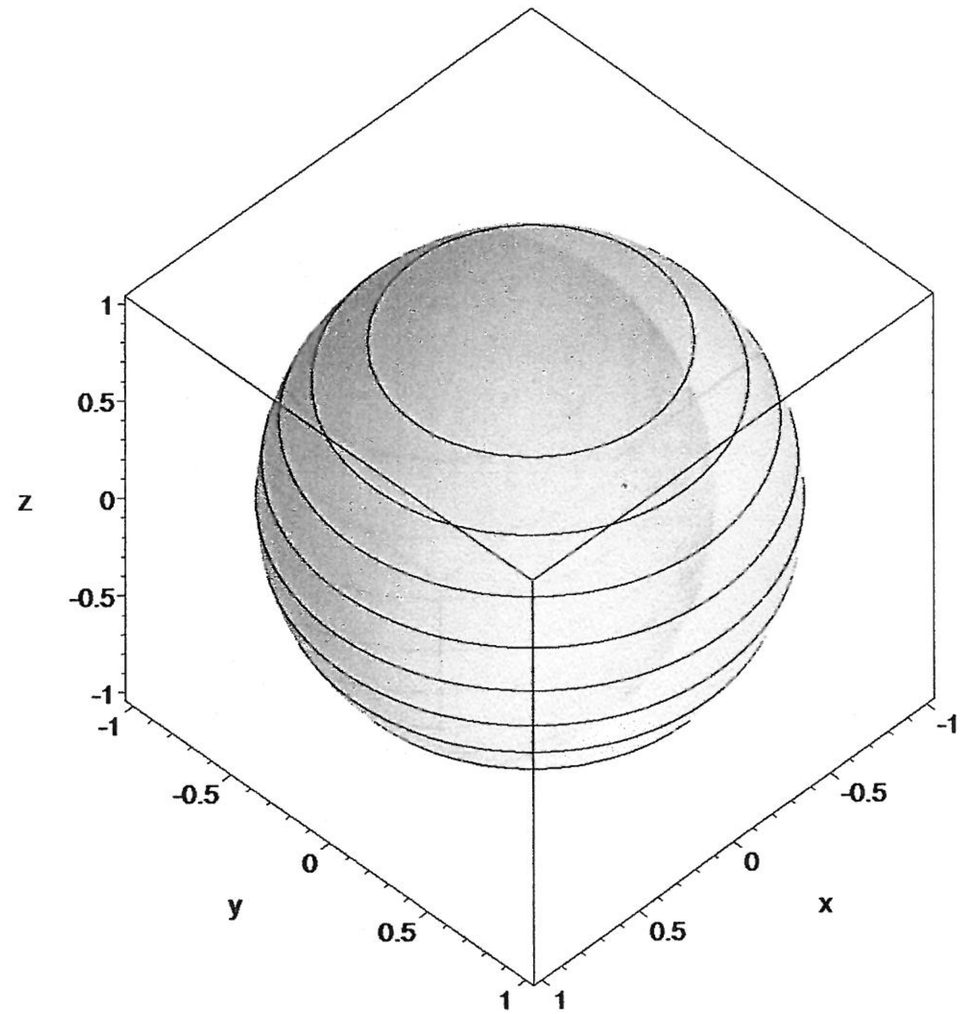
```
> plots[implicitplot]( y*x^2-2*x+1, x=-10..10, y=-10..10,
  grid=[100,100], view=[-10..10, -10..10], scaling=constrained );
```



1.3.6

We plot the polynomial equation that defines the sphere.

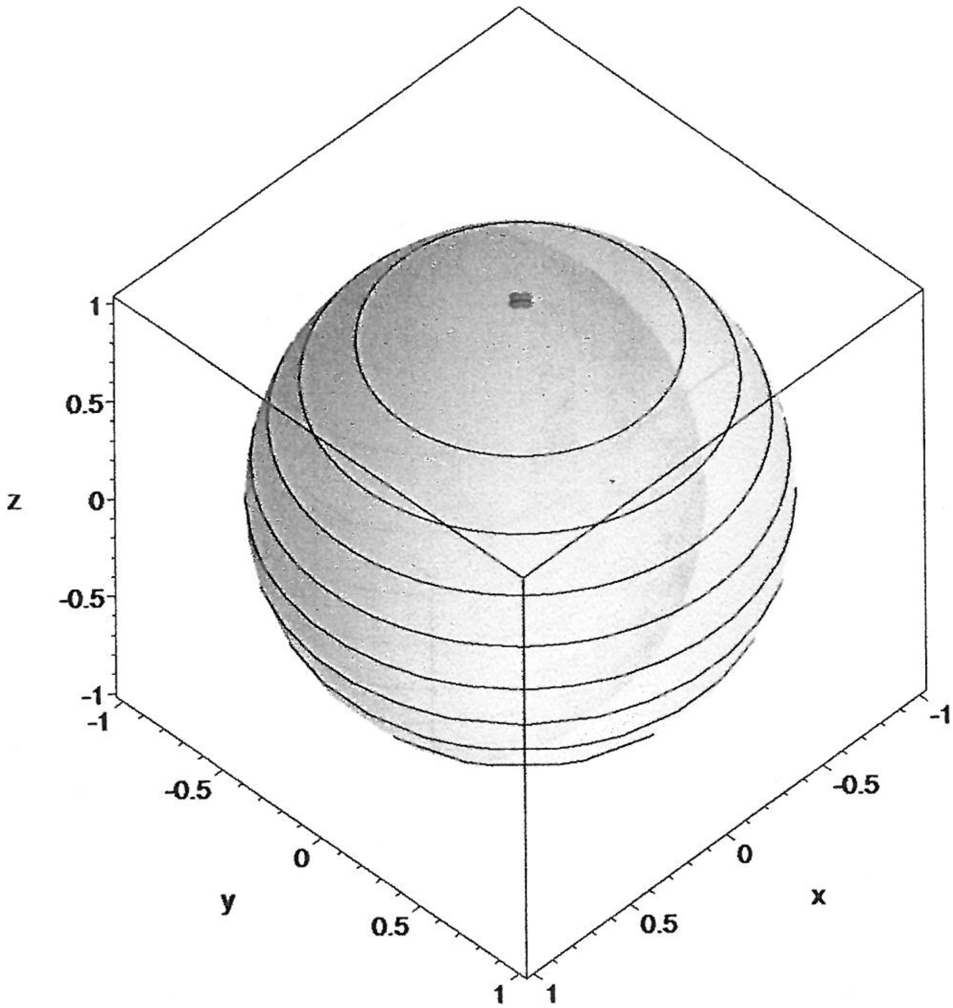
```
> plots[implicitplot3d]( x^2+y^2+z^2-1, x=-1..1, y=-1..1,
  z=-1..1, axes=box, grid=[25,25,25], style=patchcontour,
  scaling=constrained );
```



We plot the derived parametrization.

```
> plot3d( [2*s/(s^2+t^2+1), 2*t/(s^2+t^2+1),
  (s^2+t^2-1)/(s^2+t^2+1)], s=-40..40, t=-40..40, axes=box,
  labels=[x,y,z], grid=[500,500], style=patchcontour,
  scaling=constrained );
```

And note the hole in the top.



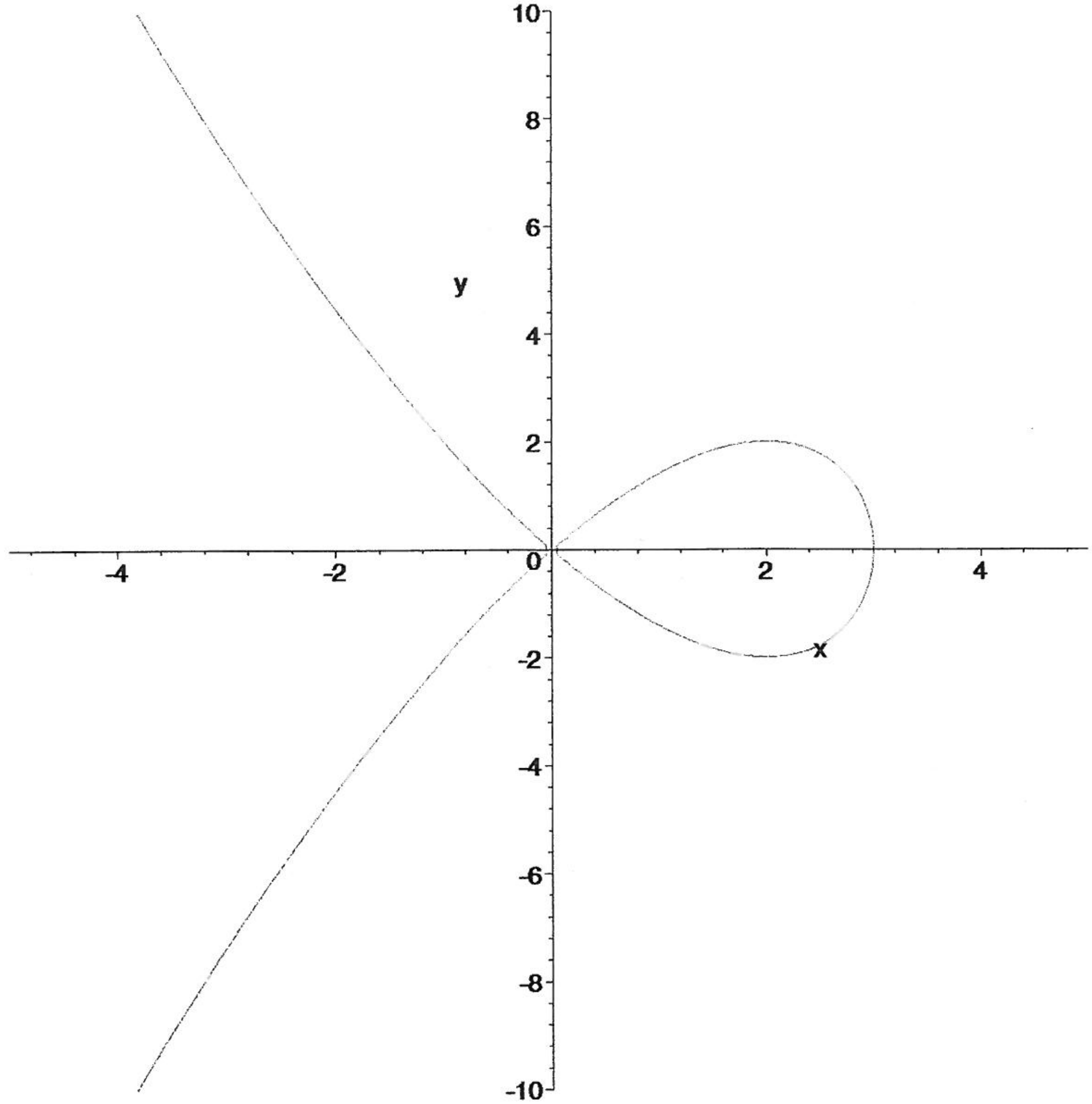
1.3.8

We choose a value for c and plot the given polynomial equation that defines the variety.

```
> c := 3;
```

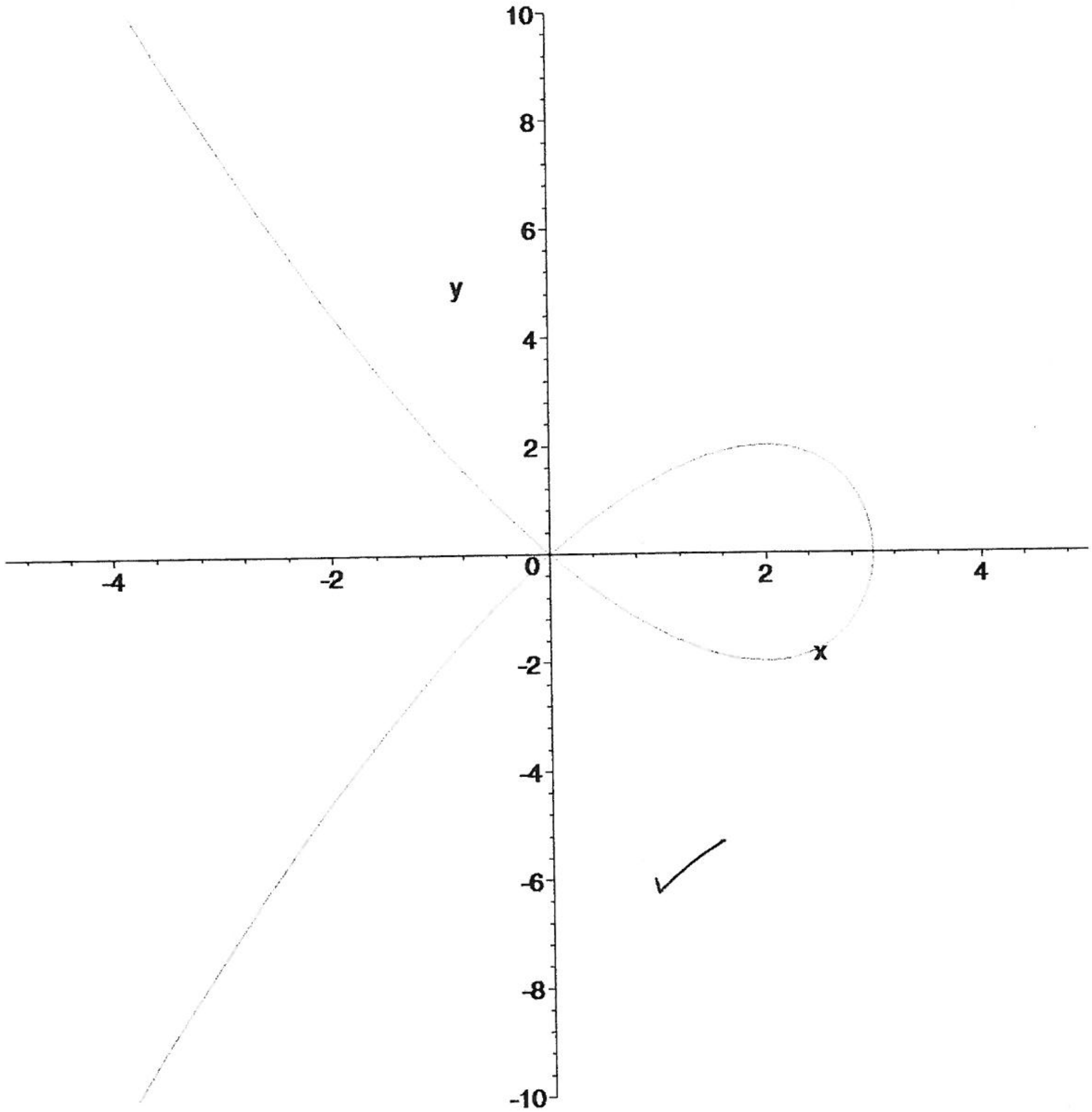
$$c := 3$$

```
> plots[implicitplot]( y^2-c*x^2+x^3, x=-5..5, y=-10..10,
  grid=[100,100], view=[-5..5, -10..10] );
```



We plot our derived parametrization.

```
> plot( [c-t^2, t*(c-t^2), t=-c..c], color=blue, view=[-5..5,
  -10..10], labels=[x,y] );
```



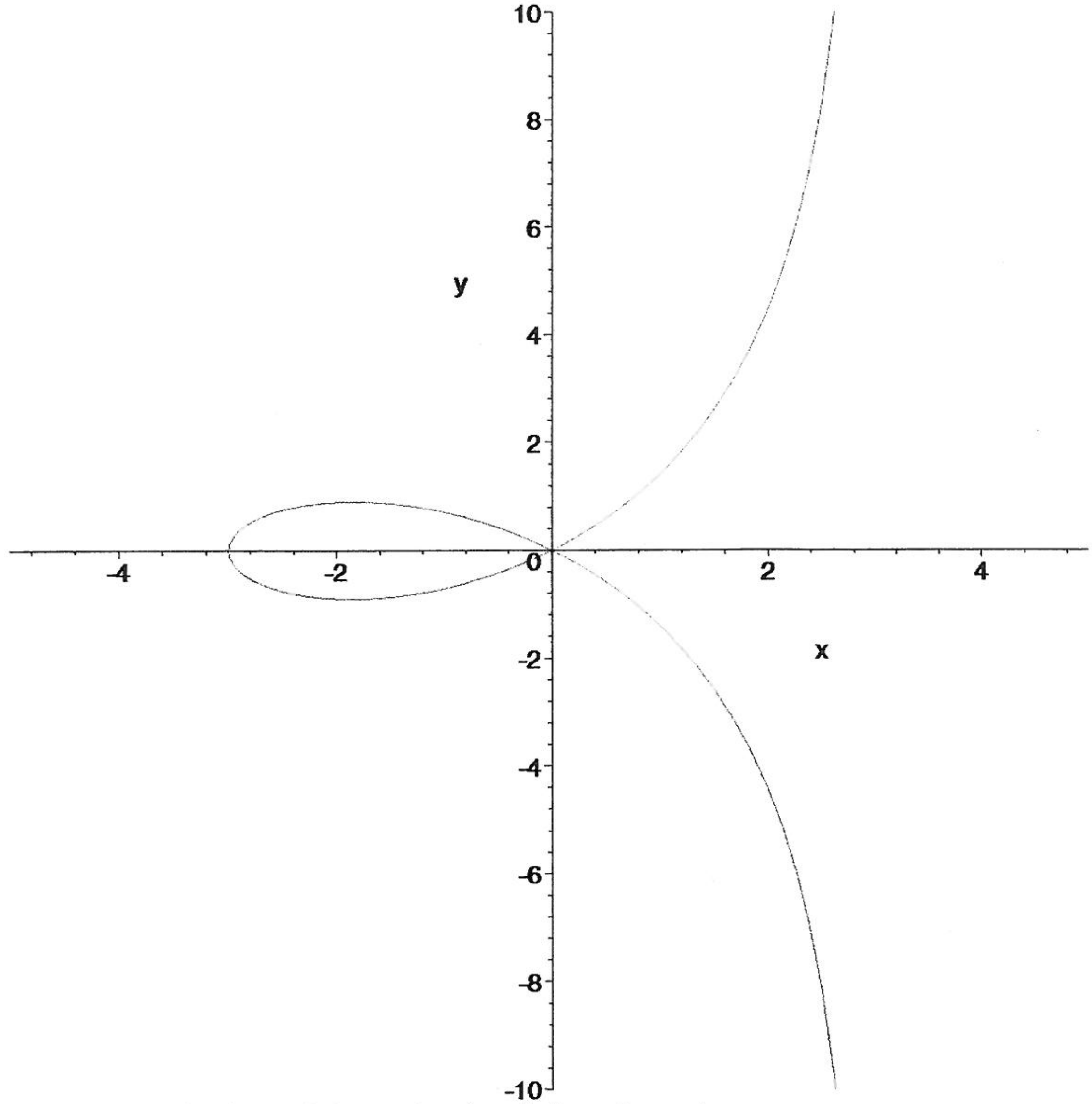
1.3.9

We choose a value for *a* and plot the given trigonometric parametrization.

```
> a := 3;
```

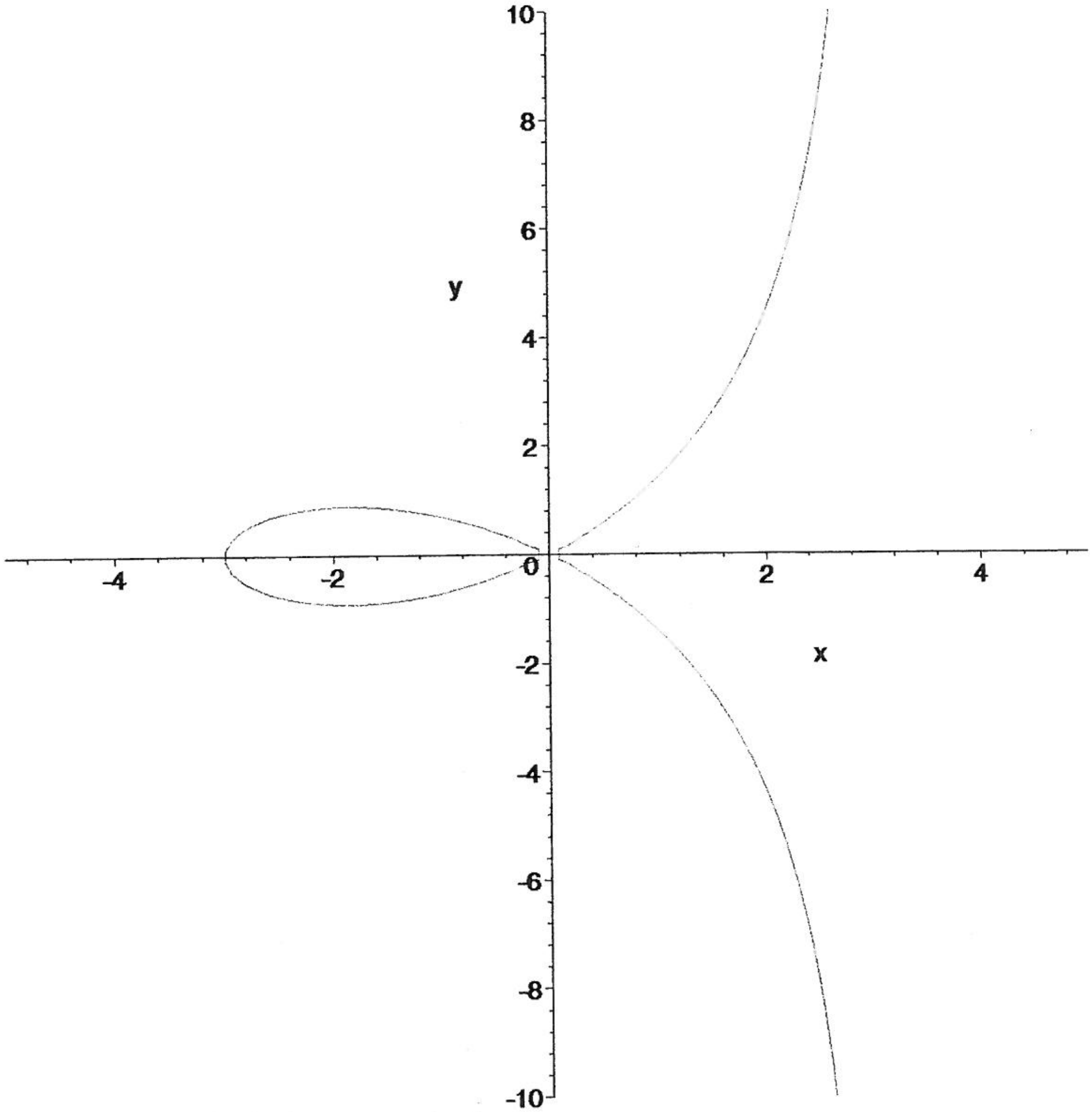
$$a := 3$$

```
> plot( [a*sin(t), a*tan(t)*(1+sin(t)), t=-3*Pi/2..Pi/2],
  color=magenta, discont=true, view=[-5..5, -10..10],
  labels=[x,y] );
```



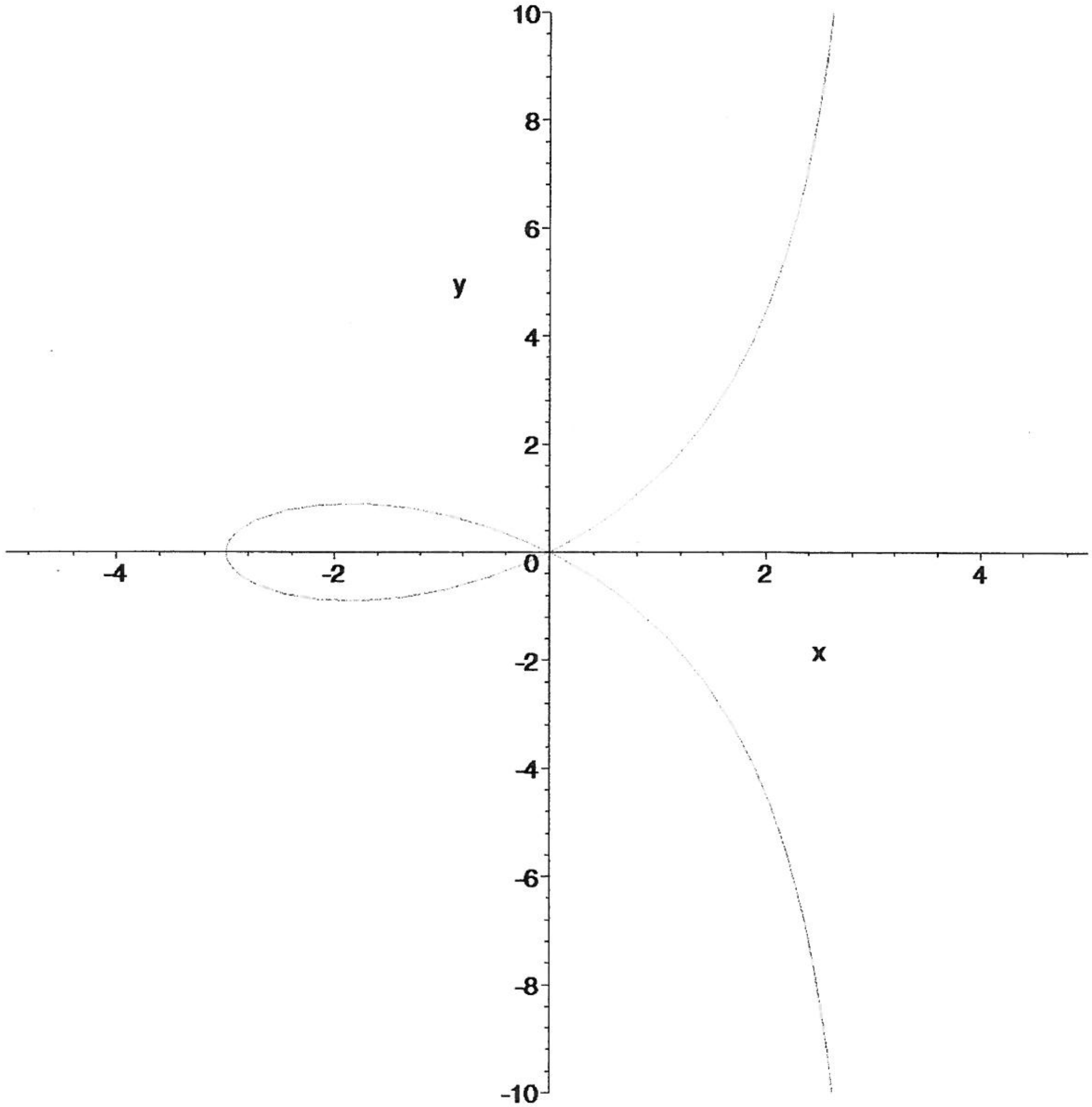
We plot our derived polynomial equation that defines the variety.

```
> plots[implicitplot]( (a-x)*y^2 = x^2*(a+x), x=-5..5, y=-10..10,
  grid=[100,100], view=[-5..5, -10..10] );
```



We plot our derived algebraic parametrization.

```
> plot( [a*(t^2-1)/(t^2+1), a*t*(t^2-1)/(t^2+1), t=-5..5],
  color=blue, view=[-5..5, -10..10], labels=[x,y] );
```



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>

MATH 800 Assignment 1
(due May 24, 2006, 9:30)

```
> restart;
```

1.3.16

```
> xc := ( (1-t)^2*x[1]+2*t*(1-t)*w*x[2]+t^2*x[3] )/(
  (1-t)^2+2*t*(1-t)*w+t^2 );
  yc := ( (1-t)^2*y[1]+2*t*(1-t)*w*y[2]+t^2*y[3] )/(
  (1-t)^2+2*t*(1-t)*w+t^2 );
```

$$xc := \frac{(1-t)^2 x_1 + 2\,t\,(1-t)\,w\,x_2 + t^2 x_3}{(1-t)^2 + 2\,t\,(1-t)\,w + t^2}$$

$$yc := \frac{(1-t)^2 y_1 + 2\,t\,(1-t)\,w\,y_2 + t^2 y_3}{(1-t)^2 + 2\,t\,(1-t)\,w + t^2}$$

a)

Show that the denominator $(1-t)^2 + 2\,t\,(1-t)\,w + t^2$ never vanishes.

Suppose that $(1-t)^2 + 2\,t\,(1-t)\,w + t^2 = 0$.

Now $0 \le (1-t)^2$ with equality iff $t = 1$, and $0 \le t^2$ with equality iff $t = 0$, so $0 < (1-t)^2 + t^2$.

Then $2\,t\,(1-t)\,w < 0$. This implies $2\,t\,(1-t) < 0$ by the assumption $0 \le w$.

The solutions to this are $t < 0$ or $1 < t$. But we are also given that $0 \le t$ and $t \le 1$.

Therefore there are no values such that $(1-t)^2 + 2\,t\,(1-t)\,w + t^2 = 0$.

We can verify these calculations in Maple as follows.

```
> solve( {denom(xc) = 0, w >= 0}, {t,w} );
```

$$\left\{w = \frac{1 - 2\,t + 2\,t^2}{2\,t\,(-1+t)},\ t < 0\right\},\ \left\{w = \frac{1 - 2\,t + 2\,t^2}{2\,t\,(-1+t)},\ 1 < t\right\}$$

b)

```
> subs( t=0, [xc, yc] );
  subs( t=1, [xc, yc] );
```

$$[x_1, y_1]$$

$$[x_3, y_3]$$

So x_1, y_1, x_3, y_3 give the coordinates of the start and end points of the curve.

c)

```
> v0 := factor( subs( t=0, diff( [xc, yc], t ) ) );
  v1 := factor( subs( t=1, diff( [xc, yc], t ) ) );
```

$$v0 := [-2\,w\,(-x_2 + x_1), -2\,w\,(-y_2 + y_1)]$$

$$v1 := [-2\,w\,(-x_3 + x_2), 2\,w\,(-y_2 + y_3)]$$

$[x'(0), y'(0)] = 2\,w\,([x_2, y_2] - [x_1, y_1])$ and $[x'(1), y'(1)] = 2\,w\,([x_3, y_3] - [x_2, y_2])$. That is, the tangent vector at the start of the curve is $2\,w$ times the vector from $[x_1, y_1]$ to $[x_2, y_2]$, and the tangent vector at the end of the curve is $2\,w$ times the vector from $[x_2, y_2]$ to $[x_3, y_3]$.

To show that $[x_2, y_2]$ is the intersection of the tangent lines at the start and end of the curve, we construct these tangent lines and compute their intersection (where they equal each other).

```
> T0 := [x[1], y[1]]+r*v0;
  T1 := [x[3], y[3]]+s*v1;
  solve( {op( expand(T0-T1) )}, {r,s} );
  expand( subs( %, [T0, T1] ) );
```

$$T0 := [x_1, y_1] + r\,[-2\,w\,(-x_2 + x_1), -2\,w\,(-y_2 + y_1)]$$

$$T1 := [x_3, y_3] + s\,[-2\,w\,(-x_3 + x_2), 2\,w\,(-y_2 + y_3)]$$

$$\{r = \frac{1}{2\,w}, s = -\frac{1}{2\,w}\}$$

$$[[x_2, y_2], [x_2, y_2]]$$

$[x_1, y_1]$, $[x_2, y_2]$, $[x_3, y_3]$ are called the control points of the curve because they define the start and end points and directions of the curve. ✓ ✓

d)

The control polygon in the triangle with vertices $[x_1, y_1]$, $[x_2, y_2]$, $[x_3, y_3]$. Because it is a triangle, it is a convex set. ✓

```
> map2( coeff, xc, [x[1], x[2], x[3]] );
  factor( add( c, c=% ) );
```

$$\left[\frac{(1-t)^2}{(1-t)^2 + 2\,t\,(1-t)\,w + t^2}, \frac{2\,t\,(1-t)\,w}{(1-t)^2 + 2\,t\,(1-t)\,w + t^2}, \frac{t^2}{(1-t)^2 + 2\,t\,(1-t)\,w + t^2}\right]$$

$$1$$ ✓

We compute the coefficients of $[x_1, y_1]$, $[x_2, y_2]$, $[x_3, y_3]$ in $[x(t), y(t)]$ (these are the same as the coefficients of x_1, x_2, x_3 in $x(t)$. Since these coeffients are all between 0 and 1 and they sum to 1, it follows that $[x(t), y(t)]$ is within the convex set containing $[x_1, y_1]$, $[x_2, y_2]$, $[x_3, y_3]$. That is, every point of the curve lies in its control polygon.

The parameter w controls the length of the tangent vectors, i.e. the velocities, at the start and end of the curve. We will investigate w further in the final 2 parts of this exercise.

e)

We evaluate $\left[x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right]$.

```
> subs( t=1/2, [xc, yc] );
  factor( % );
```

$$\left[\frac{\frac{1}{4}x_1+\frac{1}{2}w\,x_2+\frac{1}{4}x_3}{\frac{1}{2}+\frac{w}{2}}, \frac{\frac{1}{4}y_1+\frac{1}{2}w\,y_2+\frac{1}{4}y_3}{\frac{1}{2}+\frac{w}{2}}\right]$$

$$\left[\frac{1}{2}\frac{x_1+2\,w\,x_2+x_3}{1+w}, \frac{1}{2}\frac{y_1+2\,w\,y_2+y_3}{1+w}\right]$$

```
> map( collect, %, {x[2], y[2]} );
```

$$\left[\frac{w\,x_2}{1+w}+\frac{1}{2}\frac{x_1+x_3}{1+w}, \frac{w\,y_2}{1+w}+\frac{1}{2}\frac{y_1+y_3}{1+w}\right]$$

We can see that $\left[x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right] = \frac{\frac{[x_1, y_1]}{2}+\frac{[x_3, y_3]}{2}}{1+w} + \frac{w\,[x_2, y_2]}{1+w}$ as desired, but we can also verify this in Maple.

```
> expand( % - (1/(1+w)*(1/2*[x[1], y[1]]+1/2*[x[3], y[3]]) +
  w/(1+w)*[x[2], y[2]]) );
```

$$[0, 0]$$

We have shown that $\left[x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right]$ is a linear combination of $\frac{[x_1, y_1]}{2}+\frac{[x_3, y_3]}{2}$ (the midpoint of $[x_1, y_1]$ and $[x_3, y_3]$) and $[x_2, y_2]$ whose coefficients ($\frac{1}{1+w}$ and $\frac{w}{1+w}$) are between 0 and 1 and sum to 1. This shows that $\left[x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right]$ lies on the line segment connecting $[x_2, y_2]$ to the midpoint of $[x_1, y_1]$ and $[x_3, y_3]$.

f)

We define a little procedure to compute the distance squared between two points.

```
> dist2 := proc( P::list, Q::list )
    return (Q[1]-P[1])^2+(Q[2]-P[2])^2;
  end:
```

We want to show that w is the ratio $\frac{a}{b}$ where a and b are the distances from $\left[x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right]$ to the midpoint of $[x_1, y_1]$ and $[x_3, y_3]$ and to $[(x_2) . (y_2)]$, respectively. We use our distance squared procedure to compute $\frac{a^2}{b^2}$ and simplify the result.

```
> dist2( subs( t=1/2, [xc, yc] ), [x[1], y[1]]/2+[x[3], y[3]]/2
  ) / dist2( subs( t=1/2, [xc, yc] ), [x[2], y[2]] );
  factor( % );
```

$$\frac{\left(\frac{1}{2}x_1+\frac{1}{2}x_3-\frac{\frac{1}{4}x_1+\frac{1}{2}w\,x_2+\frac{1}{4}x_3}{\frac{1}{2}+\frac{w}{2}}\right)^2+\left(\frac{1}{2}y_3+\frac{1}{2}y_1-\frac{\frac{1}{4}y_1+\frac{1}{2}w\,y_2+\frac{1}{4}y_3}{\frac{1}{2}+\frac{w}{2}}\right)^2}{\left(x_2-\frac{\frac{1}{4}x_1+\frac{1}{2}w\,x_2+\frac{1}{4}x_3}{\frac{1}{2}+\frac{w}{2}}\right)^2+\left(y_2-\frac{\frac{1}{4}y_1+\frac{1}{2}w\,y_2+\frac{1}{4}y_3}{\frac{1}{2}+\frac{w}{2}}\right)^2}$$

$$w^2$$ ✓

Since a, b, w are all non-negative we can take the square root to show that $\frac{a}{b}=w$.

```
> sqrt(%) assuming w >= 0;
```

$$w$$ ✓

very nice

1.3.17

We wish to parametrize the portion of the unit circle $x^2+y^2=1$ in the first quadrant. Since this arc of the circle starts at $[1, 0]$ with a vertical tangent and ends at $[0, 1]$ with a horizontal tangent, these tangent lines will meet at $[1, 1]$. Therefore $(x_1, y_1, x_2, y_2, x_3, y_3)=(1, 0, 1, 1, 0, 1)$. ✓

```
> (P1,P2,P3) := [1, 0], [1, 1], [0, 1];
```

$$P1, P2, P3 := [1, 0], [1, 1], [0, 1]$$

The final parameter w depends on the distances defined by the midpoint of $[x_1, y_1]$ and $[x_3, y_3]$, $[x_2, y_2]$, and the point where the segment between these two intersects the curve. We compute the midpoint, the line segment, and the point where it intersects the arc of the circle.

```
> P4 := expand( P1/2+P3/2 );
```

$$P4 := \left[\frac{1}{2}, \frac{1}{2}\right]$$ ✓

```
> L := expand( t*P4+(1-t)*P2 );
  solve( subs( {x=L[1], y=L[2]}, x^2+y^2-1), {t} );
```

$$L := \left[1-\frac{t}{2}, 1-\frac{t}{2}\right]$$

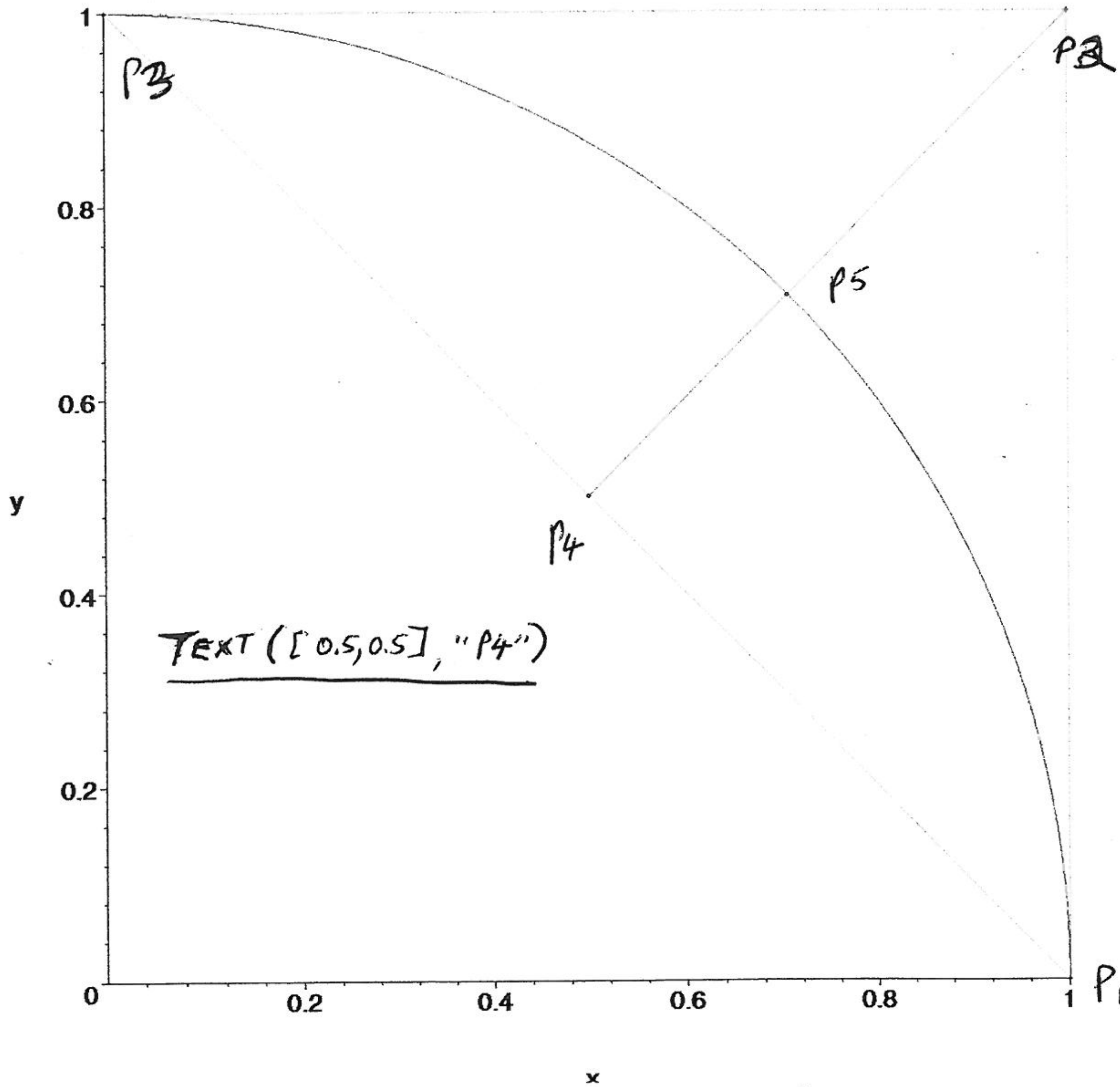
$$\{t=2+\sqrt{2}\}, \{t=2-\sqrt{2}\}$$ $0<t<1$ ✓

```
> P5 := subs( %[2], L );
```

$$P5 := \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]$$

We plot the arc of the circle (conventionally), the control points and control polygon, and the points and segments that will define w. w will be the ratio of the lengths of the maroon segment to the blue segment.

```
> C := plots[implicitplot]( x^2+y^2-1, x=0..1, y=0..1, color=red,
  scaling=constrained ):
  T := plot( [[1, t, t=0..1], [t, 1, t=0..1], [t, 1-t, t=0..1]],
  x=0..1, y=0..1, color=green, scaling=constrained ):
  L := plot( [[op(L), t=0..2-sqrt(2)], [op(L), t=2-sqrt(2)..1]],
  x=0..1, y=0..1, color=[blue,magenta], scaling=constrained ):
  P := plots[pointplot]( [P1, P2, P3, P4, P5], color=black,
  scaling=constrained ):
  plots[display]( P, L, C, T );
```



We compute *w*.

```
> factor( sqrt( dist2( P4, P5 ) / dist2( P2, P5 ) ) );
```

$$\frac{\sqrt{2}}{2}$$

V. GOOD
i.e. convincing.

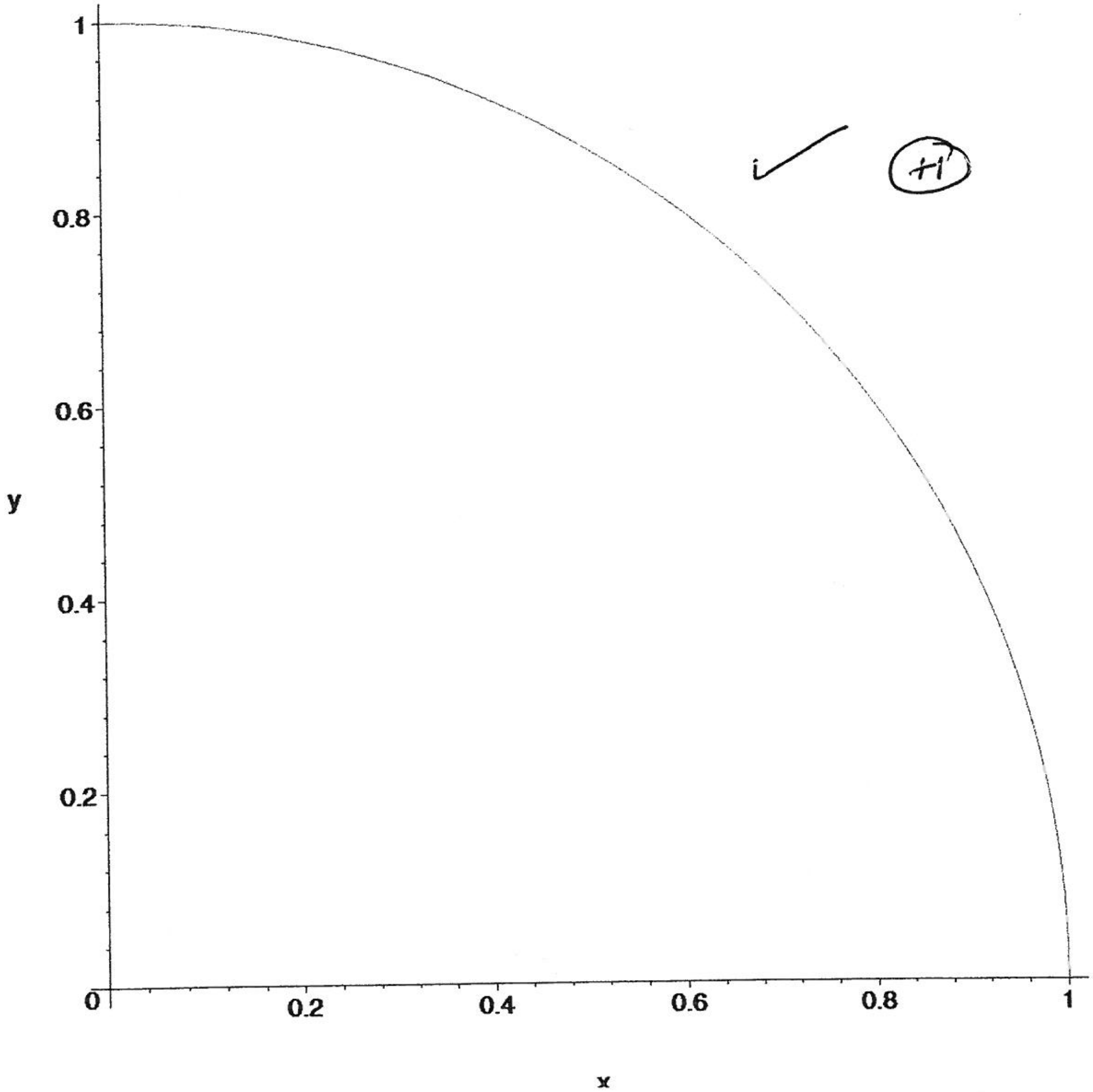
We see that $w = \frac{\sqrt{2}}{2}$, or $\frac{1}{\sqrt{2}}$, as expected.

We input the values of $x_1, y_1, x_2, y_2, x_3, y_3, w$ into $[x(t), y(t)]$ to get the parametrization of the arc of the circle, and we plot the curve from this parametrization.

```
> subs( {x[1]=1, y[1]=0, x[2]=1, y[2]=1, x[3]=0, y[3]=1,
  w=1/sqrt(2)}, [xc, yc] );
```

$$\left[\frac{(1-t)^2 + t(1-t)\sqrt{2}}{(1-t)^2 + t(1-t)\sqrt{2} + t^2}, \frac{t(1-t)\sqrt{2} + t^2}{(1-t)^2 + t(1-t)\sqrt{2} + t^2}\right]$$

```
> plot( [op(%), t=0..1], labels=[x,y], scaling=constrained );
```



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>

MATH 800 Assignment 1
(due May 24, 2006, 9:30)

```
> restart;
```

We write a procedure to recursively compute the greatest common divisor of an arbitrary number of polynomials. This will be used in subsequent exercises.

```
> GCD := proc() local f, g;
    if (nargs < 1) then return 0 fi;
    if (nargs = 1) then return args[1] fi;
    return gcd( args[1], GCD( args[2..-1] ) );
  end:
```

or gcd(args[1], args[1]) to make it canonical.

1.5.8

a)

```
> GCD( x^4+x^2+1, x^4-x^2-2*x-1, x^3-1 );
```

$$x^2+x+1$$

b)

```
> GCD( x^3+2*x^2-x-2, x^3-2*x^2-x+2, x^3-x^2-4*x+4 );
```

$$x-1$$

1.5.9

First we compute a generator of the ideal, which is a GCD of the given list of polynomials.

```
> h := GCD( x^3+x^2-4*x-4, x^3-x^2-4*x+4, x^3-2*x^2-x+2 );
```

$$h := x-2$$

Once we know that the given ideal is simply $< h >$, to test if a polynomial f is in this ideal we simply need to check that h divides f. This can be done by performing the division algorithm $f = q\,h + r$ and checking whether the remainder r is zero.

```
> rem( x^2-4, h, x );
```

$$0$$

The remainder is zero, so $f = x^2 - 4$ is in the ideal.

1.5.15

b)

```
> f := x^11-x^10+2*x^8-4*x^7+3*x^5-3*x^4+x^3+3*x^2-x-1;
```

$$f := x^{11} - x^{10} + 2\,x^8 - 4\,x^7 + 3\,x^5 - 3\,x^4 + x^3 + 3\,x^2 - x - 1$$

We compute $f_{red} = \dfrac{f}{\mathrm{GCD}(f, \mathrm{D}(f))}$ (where $\mathrm{D}(f)$ denotes the derivative of f) as described by part (a). To perform this polynomial division we use the division algorithm to compute the desired quotient, and we check that the remainder is indeed zero.

```
> f[red] := quo( f, GCD( f, diff(f,x) ), x, 'r' ); r;
```

$$f_{red} := x^5 + x^2 - x - 1$$

$$0$$

1.5.17

First we compute the GCD of the given polynomials, $h = \mathrm{GCD}(f_1, f_2)$. Note that $V(h) = V(f_1, f_2)$.

```
> h := GCD( x^5-2*x^4+2*x^2-x, x^5-x^4-2*x^3+2*x^2+x-1 );
```

$$h := x^4 - 2x^3 + 2x - 1$$

Now $I(V(h)) = < h_{red} >$. So $\{h_{red}\}$ is a basis for $I(V(f_1, f_2))$. We compute h_{red}.

```
> h[red] := quo( h, GCD( h, diff(h,x) ), x, 'r' ); r;
```

$$h_{red} := x^2 - 1$$

$$0$$

$\{x^2 - 1\}$ is a basis for this ideal.

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```
>
```