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Assignment #4 Solutions.

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MATH 800 Assignment #4

3.5.10. Compute the discriminant of the quadratic polynomial $f = ax^2+bx+c$. Explain how your answer relates to the quadratic formula, and prove that f has a multiple root iff its discriminant vanishes.

From Exercise 8, the discriminant is defined to be

$$\begin{aligned}\operatorname{disc}(f) &= \frac{(-1)^{\frac{2\cdot 1}{2}}}{a}\cdot \operatorname{Res}(f, f', x)\\ &= -\tfrac{1}{a}\operatorname{Res}(ax^2+bx+c,\ 2ax+b,\ x)\\ &= -\frac{1}{a}\cdot\det\begin{bmatrix} a & 2a & 0\\ b & b & 2a\\ c & 0 & b\end{bmatrix}\\ &= -\frac{1}{a}(ab^2-2ab^2+4a^2c)\\ &= b^2-4ac\end{aligned}$$

✓

We also compute this in Maple. [See Maple attachment.]

This discriminant is the same as what we refer to as the "discriminant" in the quadratic formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

for the roots of f.

By Proposition 8, f and f' have a nonconstant common factor

$\iff \operatorname{Res}(f,f',x)=0$

$\iff \operatorname{disc}(f)=0.$ ✓

But the only factor of $f'=2ax+b$ is $2ax+b$, so this must be the common factor up to scalar ✓

Dividing f by $2ax+b$ gives

$$\begin{array}{r} \tfrac{1}{2}x + \tfrac{b}{4a} \\ 2ax+b\,\overline{)\,ax^2+bx+c} \\ -(ax^2+\tfrac{b}{2}x) \\ \hline \tfrac{b}{2}x+c \\ -(\tfrac{b}{2}x+\tfrac{b^2}{4a}) \\ \hline \end{array}$$

So $f = ax^2+bx+c = \left(\frac{1}{2}x+\frac{b}{4a}\right)(2ax+b) + c - \frac{b^2}{4a}$
$= \frac{1}{4a}(2ax+b)(2ax+b) + \frac{1}{4a}(4ac-b^2)$

Now if $disc(f) = b^2-4ac = 0$ then the remainder $\frac{1}{4a}(4ac-b^2) = 0$.
$\Rightarrow f = \frac{1}{4a}(2ax+b)^2 \Longrightarrow f$ has a multiple root $x = -\frac{b}{2a}$. ✓

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If on the other hand f has a multiple root $x=r$ then
$f = a(x-r)^2$ (r is a root with multiplicity 2 since $deg(f)=2$)
$= ax^2 - 2arx + ar^2$
$\Longrightarrow b = -2ar,\ c = ar^2$
$\Longrightarrow disc(f) = b^2-4ac = 4a^2r^2 - 4a^2r^2 = 0$.

$\therefore f$ has a multiple root $(x = -\frac{b}{2a})$ iff $disc(f)=0$. ✓
This agrees with what we already knew from the quadratic formula.

3.5.15. Prove that if $deg(f) = l$ and $deg(g) = m$ then $Res(f,g,x) = (-1)^{lm} \cdot Res(g,f,x)$.

Proof: Let $f = \sum_{i=0}^{l} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$.
Recall that $Res(f,g,x) = det(Syl(f,g,x))$ and $Res(g,f,x) = det(Syl(g,f,x))$,
where

$$Syl(f,g,x) = \begin{bmatrix} a_l & & & b_m & & \\ \vdots & a_l & & \vdots & b_m & \\ a_0 & \vdots & \ddots\ a_l & b_0 & \vdots & \ddots\ b_m \\ & a_0 & \ddots\ \vdots & & b_0 & \vdots \\ & & a_0 & & & b_0 \end{bmatrix}_{\underbrace{\qquad\qquad}_{m}\ \underbrace{\qquad\qquad}_{l}} \text{ and } Syl(g,f,x) = \begin{bmatrix} b_m & & & a_l & & \\ \vdots & & & \vdots & & \\ b_0 & \ddots & b_m & a_0 & \ddots & a_l \\ & & \vdots & & & \vdots \\ & & b_0 & & & a_0 \end{bmatrix}_{\underbrace{\qquad\qquad}_{l}\ \underbrace{\qquad\qquad}_{m}}.$$

If we label the columns of $Syl(f,g,x)$ $A_1, \dots, A_m, B_1, \dots, B_l$
where A_j is the column containing the a_i's starting in the j^{th} entry
and B_k is the column containing the b_i's starting in the k^{th} entry,
we get $Syl(f,g,x) = [A_1 \dots A_m\ B_1 \dots B_l]$ and $Syl(g,f,x) = [B_1 \dots B_l\ A_1 \dots A_m]$. ✓
So the only difference between these matrices is a permutation of their columns. ✓
Recall from linear algebra that swapping two columns of a matrix changes the sign of the determinant. So all we need to figure out how many column swaps it takes to transform $Syl(f,g,x)$ into $Syl(g,f,x)$. ✓

If we swap A_m with B_1, then A_m with B_2, then..., then A_m with B_ℓ we transform $Syl(f,g,x) = [A_1 \dots A_m\ B_1 \dots B_\ell]$ into $[A_1 \dots A_{m-1}\ B_1 \dots B_\ell\ A_m]$ in ℓ swaps.
Swapping A_{m-1} with $B_1, B_2, \dots, B_\ell$ gets us $[A_1 \dots A_{m-2}\ B_1 \dots B_\ell\ A_{m-1}\ A_m]$ in another ℓ swaps.
4 Continuing in this fashion, we perform ℓ swaps on each A_i to move it past $B_1, \dots, B_\ell$. ✓
Since there are m of the A_i's we perform a total of ℓm swaps ✓
The result is $[B_1 \dots B_\ell\ A_1 \dots A_m] = Syl(g,f,x)$.
$\therefore Res(f,g,x) = \det(Syl(f,g,x)) = (-1)^{\ell m} \det(Syl(g,f,x)) = (-1)^{\ell m} Res(g,f,x)$. ✓ □

3.6.1. [See Maple attachment.]
3.6.3. [See Maple attachment.]

3.6.4. Suppose that $f, g \in \mathbb{C}[x]$ are polynomials of positive degree.
(a) Show that $\gamma \in \mathbb{C}$ can be written $\gamma = \alpha + \beta$, where $f(\alpha) = g(\beta) = 0$, iff the equations $f(x) = g(y-x) = 0$ have a solution with $y = \gamma$.

If $\gamma = \alpha + \beta$ where $f(\alpha) = g(\beta) = 0$, then substituting $(x,y) = (\alpha, \gamma)$ in $f(x)$ and $g(y-x)$ gives $f(\alpha) = 0$ and $g(\gamma - \alpha) = g((\alpha+\beta) - \alpha) = g(\beta) = 0$, so $(x,y) = (\alpha, \gamma)$ is a solution for $f(x) = g(y-x) = 0$ with $y = \gamma$. ✓

2 Conversely if there is a solution with $y = \gamma$ for $f(x) = g(y-x) = 0$ then this solution has the form $(x,y) = (\alpha, \gamma)$ for some $\alpha \in \mathbb{C}$.
Then $f(\alpha) = 0$ and $g(\gamma - \alpha) = 0$. Defining $\beta = \gamma - \alpha$ gives $g(\beta) = 0$ and $\gamma = \alpha + \beta$. ✓

b) Using Theorem 3, show that γ is a root of $Res(f(x), g(y-x), x)$ iff $\gamma = \alpha + \beta$ where $f(\alpha) = g(\beta) = 0$.

2 If $y = \gamma$ is a root of $Res(f(x), g(y-x), x) \in \mathbb{C}[y]$, then by Proposition 3 either $LC_x(f(x)) \in \mathbb{C}[y]$ or $LC_x(g(y-x)) \in \mathbb{C}[y]$ vanishes at $y = \gamma$, or there is an $\alpha \in \mathbb{C}$ s.t. $f(x)$ and $g(y-x)$ vanish at $(x,y) = (\alpha, \gamma)$.

But $f, g \in \mathbb{C}[x]$, so $f(x)$ contains no y's, and if $g = \sum_{i=0}^{m} b_i x^i$
then $g(y-x) = b_m (y-x)^m + \sum_{i=0}^{m-1} b_i (y-x)^i$
$= (-1)^m b_m x^m +$ (lower order terms in x).
So $LC_x(f(x))$ and $LC_x(g(y-x))$ are both constants in $\mathbb{C}$, and do not vanish at $y=\gamma$.
Thus $\exists\, \alpha \in \mathbb{C}$ s.t. the equations $f(x) = g(y-x) = 0$ vanish at $(x,y) = (\alpha, \gamma)$.
From part (a) it follows that γ can be written $\gamma = \alpha + \beta$ where $f(\alpha) = g(\beta) = 0$.

Conversely if $\gamma = \alpha + \beta$ where $f(\alpha) = g(\beta) = 0$. Then from part (a) the equations $f(x) = g(y-x) = 0$ have a solution with $y = \gamma$.
That is, there is a solution (x,y) with $y = \gamma$ where $I = \langle f(x), g(y-x) \rangle \subset \mathbb{C}[x,y]$ vanishes.
Then $Res(f(x), g(y-x), x) \in I \cap \mathbb{C}[y]$ (by Proposition 1) vanishes at $y = \gamma$, that is, $y = \gamma$ is a root of $Res(f(x), g(y-x), x)$.

c) Construct a polynomial with coefficients in $\mathbb{Q}$ which has $\sqrt{2} + \sqrt{3}$ as a root.

Let $f, g \in \mathbb{Q}[x]$ such that $f(\sqrt{2}) = g(\sqrt{3}) = 0$. So take $f = x^2 - 2$, $g = x^2 - 3$.
Since $f, g \in \mathbb{C}[x]$, it follows from part (b) that $y = \sqrt{2} + \sqrt{3}$ is a root of $R = Res(f(x), g(y-x), x) \in \mathbb{C}[y]$. Also, since $f(x)$ and $g(y-x)$ are actually in $\mathbb{Q}[x,y]$, Proposition 1 gives $R \in \mathbb{Q}[y]$.
It remains to compute R.

$$R = Res(x^2-2, (y-x)^2-3, x) = Res(x^2-2, x^2-2xy+y^2-3, x)$$

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$$= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2y & 1 \\ -2 & 0 & y^2-3 & -2y \\ 0 & -2 & 0 & y^2-3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & -2y & 1 \\ 0 & y^2-3 & -2y \\ -2 & 0 & y^2-3 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & -2y & 1 \\ -2 & 0 & y^2-3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2y & 1 \\ 0 & y^2-3 & -2y \\ 0 & -4y & y^2-1 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & y^2-1 \end{vmatrix}$$

$$= 1 \cdot [(y^2-3)(y^2-1) - (-4y)(-2y)] - 2(-1)(y^2-1) = y^4 - 4y^2 + 3 - 8y^2 + 2y^2 - 2 = y^4 - 10y^2 + 1.$$

So $R = y^4 - 10y^2 + 1 \in \mathbb{Q}[y]$ has root $y = \sqrt{2} + \sqrt{3}$,
or equivalently $R(x) = x^4 - 10x^2 + 1 \in \mathbb{Q}[x]$ has root $x = \sqrt{2} + \sqrt{3}$.

We verify this:

$$R(\sqrt{2}+\sqrt{3}) = (\sqrt{2}+\sqrt{3})^4 - 10(\sqrt{2}+\sqrt{3})^2 + 1$$
$$= (2+2\sqrt{6}+3)^2 - 10(2+2\sqrt{6}+3) + 1$$
$$= 25 + 20\sqrt{6} + 24 - 50 - 20\sqrt{6} + 1$$
$$= 0.$$

d) Modify your construction to create a polynomial whose roots are all differences of a root of f minus a root of g.

Given $f, g \in \mathbb{C}[x]$ as before, we will repeat the construction only using $-\beta$ as a root of g.

Then the statement in part (a) becomes:
$\gamma \in \mathbb{C}$ can be written $\gamma = \alpha + (-\beta) = \alpha - \beta$ where $f(\alpha) = g(-\beta) = 0$ ~~(so $g(\beta)=0$)~~ iff the equations $f(x) = g(y-x) = 0$ ~~(i.e. $f(x) = g(x-y) = 0$)~~ have a solution with $y = \gamma$.

The statement in part (b) becomes:
γ is a root of $Res(f(x), g(y-x), x)$ ~~(i.e. $Res(f(x), g(x-y), x)$)~~ iff $\gamma = \alpha + (-\beta) = \alpha - \beta$ where $f(\alpha) = g(-\beta) = 0$ ~~(so $g(\beta)=0$)~~.

So if we want to construct a polynomial whose roots are $\alpha - \beta$ for α a root of f and β a root of g, we simply compute $R = Res(f(x), g(y-x), x)$.

We demonstrate to find a polynomial with coefficients in $\mathbb{Q}$ which has $\sqrt{2}-\sqrt{3}$ as a root:
Again we start with $f = x^2-2$, $g = x^2-3$.
$R = Res(x^2-2, (-y-x)^2-3, x) = Res(x^2-2, x^2+2xy+y^2-3, x)$.
But this works out to be the same resultant as before, so we will again get $R = y^4 - 10y^2 + 1$, that also has $\sqrt{2}-\sqrt{3}$ as a root.

* 3.6.6. Suppose $f, g \in \mathbb{Q}[x]$ are polynomials of positive degree.

a) Describe an algorithm for determining when f and g have roots that differ by an integer.

From Exercise 4, part (d), we know that the roots of $R = \mathrm{Res}(f(x), g(-y-x), x)$ are exactly the differences in the roots of f and g. So to determine whether f and g have roots that differ by an integer, we simply compute R and check if it has any integer roots.

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We can check if R has integer roots as follows.
R has coefficients in $\mathbb{Q}$, so we can scale it so that all coefficients of R are integers. Then if R has an integer root m then m must divide the constant coefficient of R.
So if we can factor the constant coefficient, we simply need to evaluate R at each divisor of the constant coefficient to see if R vanishes there.
Note that if we find any root m we can divide $R(y)$ by its factor $(y-m)$ and continue working with the smaller polynomial.

Boo! A Terrible idea!

b) [See Maple attachment.] 2

3.6.7. [See Maple attachment.]

4.1.1. Recall that $\mathbb{V}(y-x^2, z-x^3)$ is the twisted cubic in $\mathbb{R}^3$.
a) Show that $\mathbb{V}((y-x^2)^2+(z-x^3)^2)$ is also the twisted cubic.

Let $(a,b,c) \in \mathbb{V}(y-x^2, z-x^3) \subset \mathbb{R}^3$.
Then $b-a^2=0$ and $c-a^3=0$
$\Rightarrow (b-a^2)^2+(c-a^3)^2 = 0+0=0$
$\Rightarrow (a,b,c) \in \mathbb{V}((y-x^2)^2+(z-x^3)^2)$. ✓

Now let $(a,b,c) \in \mathbb{V}((y-x^2)^2+(z-x^3)^2) \subset \mathbb{R}^3$.
Then $(b-a^2)^2+(c-a^3)^2=0$
$\Rightarrow (b-a^2)^2=0$ and $(c-a^3)^2=0$ since $(b-a^2)^2 \geq 0$ and $(c-a^3)^2 \geq 0$
2 $\Rightarrow b-a^2=0$ and $c-a^3=0$
$\Rightarrow (a,b,c) \in \mathbb{V}(y-x^2, z-x^3)$. ✓

$\therefore \mathbb{V}((y-x^2)^2+(z-x^3)^2) = \mathbb{V}(y-x^2, z-x^3)$ since ✓ we have shown inclusion in both directions.

b) Show that any variety $\mathbb{V}(I) \subset \mathbb{R}^n$, $I \subset \mathbb{R}[x_1,\dots,x_n]$, can be defined by a single equation (and, hence, by a principle ideal).

WLOG we can assume that I is an ideal.
(If not, we have $\mathbb{V}(I)=\mathbb{V}(\langle I \rangle)$, thus introducing an ideal)
Then $\exists f_1,\dots,f_s \in \mathbb{R}[x_1,\dots,x_n]$ s.t. $I = \langle f_1,\dots,f_s \rangle$. (HBT) ✓
2 So $\mathbb{V}(I) = \mathbb{V}(f_1,\dots,f_s) = \{a \in \mathbb{R}^n : f_i(a)=0 \ \forall 1 \leq i \leq s\}$. ✓
Now consider $f = \sum_{i=1}^{s} f_i^2 \in \mathbb{R}[x_1,\dots,x_n]$.
If $f_1,\dots,f_s$ vanish at a point a, then clearly f also vanishes at a. ✓
If f vanishes at a point a, then
$(\sum_{i=1}^{s} f_i^2)(a) = 0$
$\Rightarrow \sum_{i=1}^{s} f_i(a)^2 = 0$
$\Rightarrow f_i(a)^2 = 0 \ \forall 1 \leq i \leq s$ (since $f_i(a)^2 \geq 0 \ \forall a \in \mathbb{R}^n$)
$\Rightarrow f_i(a) = 0 \ \forall 1 \leq i \leq s$
so $f_1,\dots,f_s$ also vanish at a. Note that this only holds because we are working over $\mathbb{R}$, not $\mathbb{C}$. ✓
$\therefore \mathbb{V}(f_1,\dots,f_s) = \mathbb{V}(f)$, that is, $\mathbb{V}(I) = \mathbb{V}(\langle f \rangle)$. ✓
← a principle ideal.

4.1.2. Let $J = \langle x^2+y^2-1, y-1\rangle$. Find $f \in \mathbb{I}(\mathbb{V}(J))$ such that $f \notin J$.

$\mathbb{V}(J)$ is the set of all points where $x^2+y^2-1=0$ and $y-1=0$.
$\Rightarrow y=1 \Rightarrow x^2+y^2-1 = x^2+1-1 = x^2 = 0 \Rightarrow x=0$
so $\mathbb{V}(J) = \{(0,1)\}$. ✓
Now $f = x$ vanishes on $\mathbb{V}(J)$, so $f \in \mathbb{I}(\mathbb{V}(J))$. ✓

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To show that $f \notin J$, note that $G = \{x^2+y^2-1, y-1\}$ is a GB for J w.r.t. lex order $(x>y)$:

$$\begin{aligned} S(x^2+y^2-1, y-1) &= y(x^2+y^2-1) - x^2(y-1) \\ &= x^2+y^3-y \\ &= 1\cdot(x^2+y^2-1) + (y^2-1)(y-1) \\ &\to_G 0. \end{aligned}$$

✓

Then $\overline{f}^G = x \neq 0$ since $x \notin LT(G)$.
$\therefore f \notin J$.

4.1.5. Establish that $\tilde{I}$ as defined in the proof of the Weak Nullstellensatz is an ideal of $k[\tilde{x}_1,\dots,\tilde{x}_n]$.

Proof: $\tilde{I} = \{\tilde{f} : f \in I\}$ where I is an ideal of $k[x_1,\dots,x_n]$ and $\tilde{f} = f(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1, \dots, \tilde{x}_n+a_n\tilde{x}_1)$, that is, $\sim$ is the linear transformation $x_1 = \tilde{x}_1,\ x_2 = \tilde{x}_2+a_2\tilde{x}_1, \dots, x_n = \tilde{x}_n+a_n\tilde{x}_1$.

(i) $0 \in I \Rightarrow \tilde{0} = 0 \in \tilde{I}$ (note that constants are unaffected by $\sim$). ✓

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(ii) Let $g_1, g_2 \in \tilde{I}$. So $\exists f_1, f_2 \in I$ s.t. $g_1 = \tilde{f}_1$ and $g_2 = \tilde{f}_2$.

Then
$$\begin{aligned} \widetilde{(f_1+f_2)} &= (f_1+f_2)(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1, \dots, \tilde{x}_n+a_n\tilde{x}_1) \\ &= f_1(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1,\dots,\tilde{x}_n+a_n\tilde{x}_1) + f_2(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1,\dots,\tilde{x}_n+a_n\tilde{x}_1) \\ &= \tilde{f}_1 + \tilde{f}_2 = g_1+g_2. \end{aligned}$$

✓

So $g_1+g_2 = \widetilde{(f_1+f_2)} \in \tilde{I}$ because $f_1+f_2 \in I$.

(iii) Let $g \in \tilde{I}$ and $h \in k[\tilde{x}_1,\dots,\tilde{x}_n]$. So $\exists f \in I$ s.t. $g = \tilde{f}$.
Now let $h_1 = h(x_1, x_2-a_2x_1, \dots, x_n-a_nx_1) \in k[x_1,\dots,x_n]$.
Then $h_1 f \in I$ since I is an ideal, and

NICE

$$\begin{aligned} \widetilde{(h_1f)} &= (h_1f)(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1,\dots,\tilde{x}_n+a_n\tilde{x}_1) \\ &= h_1(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1,\dots,\tilde{x}_n+a_n\tilde{x}_1)\cdot f(\tilde{x}_1, \tilde{x}_2+a_2\tilde{x}_1,\dots,\tilde{x}_n+a_n\tilde{x}_1) \\ &= h(\tilde{x}_1, (\tilde{x}_2+a_2\tilde{x}_1)-a_2\tilde{x}_1, \dots, (\tilde{x}_n+a_n\tilde{x}_1)-a_n\tilde{x}_1)\cdot\tilde{f} = h(\tilde{x}_1,\tilde{x}_2,\dots,\tilde{x}_n)\, g. \end{aligned}$$

✓

So $hg = \widetilde{(h_1f)} \in \tilde{I}$ because $h_1f \in I$. $\therefore \tilde{I}$ is an ideal. $\square$

* 4.1.10. For the two ideals in $\mathbb{R}[x,y,z]$ from Exercise 1 that give the same nonempty variety, show that one is contained in the other. Can you find two ideals in $\mathbb{R}[x,y]$, neither contained in the other, which give the same nonempty variety? Can you do the same for $\mathbb{R}[x]$?

i) In Exercise 1 we saw that $\mathbb{V}(\langle y-x^2, z-x^3\rangle) = \mathbb{V}(\langle (y-x^2)^2+(z-x^3)^2\rangle)$, the twisted cubic.
Now $(y-x^2)^2+(z-x^3)^2 = A(y-x^2)+B(z-x^3)$ where $A=y-x^2$, $B=z-x^3$,
so $(y-x^2)^2+(z-x^3)^2 \in \langle y-x^2, z-x^3\rangle$. ✓
$\therefore \langle (y-x^2)^2+(z-x^3)^2\rangle \subset \langle y-x^2, z-x^3\rangle$. ✓

ii) Let $I_1 = \langle x^2, y\rangle$, $I_2 = \langle x, y^2\rangle \in \mathbb{R}[x,y]$.
Then $\mathbb{V}(I_1) = \mathbb{V}(I_2) = \{(0,0)\}$. ✓
$x \in I_2$ but $x \notin I_1 = \{Ax^2+By\}$ since every term T of Ax^2+By has either $\deg_x T \geq 2$ or $\deg_y T \geq 1$. ~~So $I_2 \not\subset I_1$.~~ or $Ax^2+By=0$.
Similarly $y \in I_1$ but $y \notin I_2$, so $I_1 \not\subset I_2$. ✓

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iii) In $R[x]$, every ideal is a principal ideal. So for $I_1 = \langle g_1\rangle$ and $I_2 = \langle g_2\rangle$ to give the same nonempty variety but neither ideal contained in the other, we need g_1 and g_2 to have the same roots but have neither polynomial be a multiple of the other.
For example, let $g_1 = x^3-x^2 = x^2(x-1)$ and $g_2 = x^3-2x^2+x = x(x-1)^2$.
Then $\mathbb{V}(I_1) = \mathbb{V}(I_2) = \{0,1\}$.
But $g_2 \nmid g_1$ and $g_1 \nmid g_2$, so $I_2 \not\subset I_1$ and $I_1 \not\subset I_2$. ✓

4.2.2. Let f and g be distinct nonconstant polynomials in $k[x,y]$ and let $I = \langle f^2, g^3\rangle$. Is it necessarily true that $\sqrt{I} = \langle f, g\rangle$.

3

No. For instance, let $f = x^3$ and $g = y^2$. Then $I = \langle x^6, y^6\rangle$ and $\sqrt{I} = \langle x, y\rangle$. While it is certainly true that $\langle f,g\rangle = \langle x^3, y^2\rangle \subset \sqrt{I}$, we have $x, y \in \sqrt{I}$ but $x, y \notin \langle f,g\rangle$ ✓ so $\sqrt{I} \neq \langle f,g\rangle$. ✓

4.2.5. Prove that $\mathbb{I}$ and $\mathbb{V}$ are inclusion-reversing.

Proof: (For $\mathbb{I}$).
Let $V_1, V_2 \subset k^n$ be varieties with $V_1 \subset V_2$.
If $f \in k[x_1, \dots, x_n]$ is in $\mathbb{I}(V_2)$ then f vanishes on V_2.
Then f vanishes on V_1, so $f \in \mathbb{I}(V_1)$.
$\therefore \mathbb{I}(V_2) \subset \mathbb{I}(V_1)$. * This is the proof of §1.4 Prop. 8(i).

(For $\mathbb{V}$).
Let $I_1, I_2 \subset k[x_1, \dots, x_n]$ be ideals with $I_1 \subset I_2$.
4 If $a \in k^n$ is in $\mathbb{V}(I_2)$ then every $f \in I_2$ vanishes at a.
Then every $f \in I_1$ vanishes at a, so $a \in \mathbb{V}(I_1)$. ✓
$\therefore \mathbb{V}(I_2) \subset \mathbb{V}(I_1)$. □

4.2.7 [See Maple attachment.]

4.2.12 [See Maple attachment.]

4.2.14. Let $J = \langle xy, (x-y)x \rangle$. Describe $\mathbb{V}(J)$ and show that $\sqrt{J} = \langle x \rangle$.

$J = \langle f_1, f_2 \rangle$ where $f_1 = xy$ and $f_2 = x^2 - xy$. Let $f_3 = f_1 + f_2 = x^2 \in J$.
Then $f_2 = f_3 - f_1 \in \langle f_1, f_3 \rangle$, so $J = \langle f_1, f_3 \rangle = \langle xy, x^2 \rangle$. ✓
Then $\mathbb{V}(J)$ contains all solutions of the system $\{xy = 0, x^2 = 0\}$.
$x^2 = 0 \Rightarrow x = 0$, then $x = 0 \Rightarrow xy = 0 \ \forall y$, so y can be anything. ✓
2 Thus $\mathbb{V}(J) = \{(0, a) : a \in k\}$. ✓

Now $x^2 \in J \Rightarrow x \in \sqrt{J}$, so $\langle x \rangle \subset \sqrt{J}$. ✓
Also, $\sqrt{J} \subset \mathbb{I}(\mathbb{V}(J))$ because $f \in \sqrt{J}$ implies $f^m \in I$ for some m, so
f^m vanishes on $\mathbb{V}(J)$, which implies f vanishes on $\mathbb{V}(J)$, thus $f \in \mathbb{I}(\mathbb{V}(J))$.
2 But $\mathbb{I}(\mathbb{V}(J)) = \langle x \rangle$, so $\sqrt{J} \subset \langle x \rangle$. ✓
$\therefore \sqrt{J} = \langle x \rangle$.

Additional exercise on resultants:

Consider $f = x^2 + 2y^2 - 3$ and $g = x^2 + xy + y^2$, and let $I = \langle f, g \rangle$.

a) Compute $R = Res(f, g, x)$ (i) using Maple, (ii) by hand.

i) [See Maple attachment.]

(ii)

$$Syl(f, g, x) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & y & 1 \\ 2y^2-3 & 0 & y^2 & y \\ 0 & 2y^2-3 & 0 & y^2 \end{bmatrix}.$$

Then

$$\begin{aligned} Res(f,g,x) &= \det(Syl(f,g,x)) \\ &= 1 \cdot \begin{vmatrix} 1 & y & 1 \\ 0 & y^2 & y \\ 2y^2-3 & 0 & y^2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 2y^2-3 & 0 & y \\ 0 & 2y^2-3 & y^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & \boxed{y} & 1 \\ -y & \boxed{0} & 0 \\ 2y^2-3 & \boxed{0} & y^2 \end{vmatrix} + (-1)(2y^2-3) \begin{vmatrix} 1 & 1 \\ 2y^2-3 & y^2 \end{vmatrix} \\ &= y(y^3) - (2y^2-3)(y^2-2y^2+3) \\ &= y^4 + 2y^4 - 9y^2 + 9 \\ &= 3y^4 - 9y^2 + 9. \end{aligned}$$

2

b) Is $\langle R \rangle = I \cap \mathbb{Q}[y]$?

1 [See Maple attachment.]

c) Find $A, B \in \mathbb{Q}[x,y]$ s.t. $Af + Bg = R$.

2 [See Maple attachment.]

Additional exercise on Buchberger's algorithm:

[See Maple attachment.]

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```
> restart;
```

3.5.10

We enter the polynomial f and compute its discriminant.

```
> f := a*x^2+b*x+c;
  R := resultant( f, diff(f,x), x );
  disc := simplify(
  (-1)^(degree(f,x)*(degree(f,x)-1)/2)/lcoeff(f,x)*R );
```

$$f := a\,x^2 + b\,x + c$$

$$R := 4\,a^2\,c - b^2\,a$$

$$disc := -4\,a\,c + b^2$$

✓

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```
>
```

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```
> restart;
```

3.6.1

3

We enter the given polynomials f and g.

```
> f := x^2*y-3*x*y^2+x^2-3*x*y;
  g := x^3*y+x^3-4*y^2-3*y+1;
```

$$f := x^2 y - 3 x y^2 + x^2 - 3 x y$$
$$g := x^3 y + x^3 - 4 y^2 - 3 y + 1$$

We compute Res(f, g, x) and Res(f, g, y).

```
> resultant( f, g, x );
  resultant( f, g, y );
```

$$-(-4 y^2 - 3 y + 1)(y + 1)^4 (4 y - 1 - 27 y^3)$$
$$0$$

Since Res(f, g, y) = 0, by Proposition 1, f and g must have a common factor with degree at least 1 in y. Also since Res(f, g, x) $\neq 0$, f and g have no common factor containing x. We verify this by computing the gcd.

```
> gcd(f,g);
```

$$y + 1$$

3.6.3

a)

We enter f and g.

```
> f := x*y-1;
  g := x^2+y^2-4;
```

$$f := x y - 1$$
$$g := x^2 + y^2 - 4$$

We compute Res(f, g, x) $\in$ $(I \cap k[y])$, where I is generated by $\{f, g\}$.

```
> resultant( f, g, x );
```

$$-4 y^2 + y^4 + 1$$

We compute a Groebner basis G for I w.r.t. lexicographic order, $y < x$. By the Elimination Theorem, $G \cap k[y]$ is a Groebner basis for $I \cap k[y]$.

```
> Groebner[Basis]( [f,g], plex(x,y) );
```

$$[-4 y^2 + y^4 + 1, -4 y + y^3 + x]$$

Since $G \cap k[y] = \{\text{Res}(f, g, x)\}$, the resultant does generate the elimination ideal $I \cap k[y]$.

b)

We enter f and g.

```
> f := x*y-1;
  g := y*x^2+y^2-4;
```

$$f := x\,y - 1$$

$$g := x^2\,y + y^2 - 4$$

We compute $\mathrm{Res}(f, g, x) \in (I \cap k[y])$, where I is generated by $\{f, g\}$.

```
> resultant( f, g, x );
```

$$-4\,y^2 + y^4 + y$$

We compute a Groebner basis G for I w.r.t. lexicographic order, $y < x$. By the Elimination Theorem, $G \cap k[y]$ is a Groebner basis for $I \cap k[y]$.

```
> Groebner[Basis]( [f,g], plex(x,y) );
```

$$[1 + y^3 - 4\,y, x + y^2 - 4]$$

3

Since $G \cap k[y] \neq \{\mathrm{Res}(f, g, x)\}$, the resultant does not generate the elimination ideal $I \cap k[y]$. Instead $< \mathrm{Res}(f, g, x) > \subseteq (I \cap k[y])$.

There seems to be some connection between the resultants in $I \cap k[y]$ and the Extension Theorem regarding partial solutions in $V(I \cap k[y])$.

Note that in part (a) the leading coefficient (w.r.t. x) of at least one of the generators $\{f, g\}$ of I was constant. Thus the Extension Theorem guarantees that any partial solution in $V(I \cap k[y])$ will extend to a solution in $V(I)$. Also notice that here we ended up with $\mathrm{Res}(f, g, x)$ generating all of $I \cap k[y]$.

Now in part (b) the leading coefficients (w.r.t. x) of both generators $\{f, g\}$ of I were multiples of y, hence vanished at $y = 0$. Thus if $y = 0$ is a partial solution, the Extension Theorem does not guarantee that it will extend. In fact, we see from the Groebner basis that $y = 0$ is not a partial solution. Now notice that here we ended up with $\mathrm{Res}(f, g, x)$ consisting of the generator for $I \cap k[y]$ with the extra factor $y - 0$ tacked on. This does not seem to be a coincidence.

3.6.6

We enter the given polynomials in function form.

```
> f := x -> x^5-2*x^3-2*x^2+4;
  g := x -> x^5+5*x^4+8*x^3+2*x^2-5*x+1;
```

$$f := x \to x^5 - 2\,x^3 - 2\,x^2 + 4$$

$$g := x \to x^5 + 5\,x^4 + 8\,x^3 + 2\,x^2 - 5\,x + 1$$

We compute $\mathrm{Res}(\mathrm{f}(x), \mathrm{g}(x - y), x)$. We know that this will give a polynomial in y whose roots will be the difference of a root of f minus a root of g.

```
> R := resultant( f(x), g(-y+x), x );
```

$$R := 591325 + 4330965\,y + 16975312\,y^6 + 9718360\,y^7 - 338080\,y^{14} - 260176\,y^{15} - 1286472\,y^{11} - 896052\,y^{13} - 1726724\,y^{12} + 13939388\,y^2 + 26059084\,y^3 + 7746\,y^{21} + 280\,y^{23} + 25\,y^{24} + y^{25} + 1840\,y^{22} + 4940\,y^{18} + 32220\,y^{19} + 20986\,y^{20} - 96861\,y^{17} + 4512519\,y^9$$

$+ 1355928\,y^{10} - 222853\,y^{16} + 6606927\,y^8 + 31657034\,y^4 + 26744434\,y^5$

We compute the rational roots of this polynomial.

```
> roots( R, y );
```

$$[[-1, 5]]$$

We see that 1 -1 is a root with multiplicity 5. That means there are 5 pairs consisting of a root α of f and a root β of g that have a difference $\alpha - \beta = 1$. We can verify this by factoring f and g (completely over C).

```
> factor( f(x), complex );
  factor( g(x), complex );
```

$$(x + 1.414213562)\,(x + 0.6299605249 + 1.091123636\,I)\,(x + 0.6299605249 - 1.091123636\,I)\,(x - 1.259921050)\,(x - 1.414213562)$$

$$(x + 2.414213562)\,(x + 1.629960525 + 1.091123636\,I)\,(x + 1.629960525 - 1.091123636\,I)\,(x - 0.2599210499)\,(x - 0.4142135624)$$

4

3.6.7

We enter the polynomials f and g from the given parametric equations.

```
> f := u*(1+t^2)-t^2;
  g := v*(1+t^2)-t^3;
```

$$f := u\,(1 + t^2) - t^2$$
$$g := v\,(1 + t^2) - t^3$$

We compute Res(f, g, t) to eliminate t. The parametrized curve is then described by the implicit equation Res(f, g, t) = 0.

```
> resultant( f, g, t );
```

2

$$u^3 + v^2\,u - v^2$$

To check this against our previous method of implicitization, we compute a Groebner basis G for the ideal generated by f, g w.r.t. lex order, $(u, v) < t$. Note that since the denomator $1 + t^2$ is relatively prime to the numerators, we don't have to include the extra polynomial to guarantee that $1 + t^2$ won't vanish.

```
> Groebner[Basis]( [f,g], plex(t,u,v) );
```

$$[u^3 + v^2\,u - v^2,\ v\,t - u^2 - v^2,\ u\,t - v,\ t^2 - u^2 - u - v^2]$$

We know that the single polynomial in $G \cap k[u, v]$ defines the implicit equation for the curve. This agrees with the equation Res(f, g, t) = 0.

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>

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```
> restart;
```

4.2.7

a)

We enter the given f and the generators for the ideal I.

```
> f := x+y;
  F := expand( [x^3, y^3, x*y*(x+y)] );
```

$$f := x+y$$

$$F := [x^3, y^3, x^2 y + x y^2]$$

To determine if $f \in \sqrt{I}$, we compute a reduced Groebner basis for the ideal generated by the generators of I and the polynomial $1-tf$.

```
> Groebner[Basis]( [op(F), 1-t*f], tdeg(x,y,t) );
```

$$[1]$$

4

Since this ideal is generated by 1 (so is all of $k[x, y, t]$), Proposition 8 tells us that $f \in \sqrt{I}$.

To determine the smallest m such that $f^m \in I$, we compute a Groebner basis G for I and then divide successive powers f^i by G until one of them reduces to zero. Then this one is in I.

```
> G := Groebner[Basis]( F, tdeg(x,y) );
```

$$G := [y^3, x^2 y + x y^2, x^3]$$

r:=

```
> Groebner[NormalForm]( f, G, tdeg(x,y) );
  for i while (% <> 0) do
  r:=Groebner[NormalForm]( expand( f^(i+1) ), G, tdeg(x,y) );
  od;
  'i' = i;
```

Don't use %.

$$x+y$$

$$x^2+2xy+y^2$$

$$0$$

$$i=3$$

So f^3 is the smallest power of f that is in I.

b)

We enter the given f and the generators for the ideal I.

```
> f := x^2+3*x*z;
  F := [x+z, x^2*y, x-z^2];
```

$$f := x^2+3xz$$

$$F := [x+z, x^2 y, x-z^2]$$

To determine if $f \in \sqrt{I}$, we compute a reduced Groebner basis for the ideal generated by the generators of I and the polynomial $1 - tf$.

```
> Groebner[Basis]( [op(F), 1-t*f], tdeg(x,y,z,t) );
```

$$[1 + 2\,t, z + 1, y, x - 1]$$

Since this ideal is not generated by 1 (so is not all of $k\,[x, y, t]$), Proposition 8 tells us that **not** $f \in \sqrt{I}$.

4.2.12

We enter the given polynomial $f \in Q\,[x, y]$.

```
> f :=
x^5+3*x^4*y+3*x^3*y^2-2*x^4*y^2+x^2*y^3-6*x^3*y^3-6*x^2*y^4+x^3
*y^4-2*x*y^5+3*x^2*y^5+3*x*y^6+y^7;
```

$$f := x^5 + 3\,x^4\,y + 3\,x^3\,y^2 - 2\,x^4\,y^2 + x^2\,y^3 - 6\,x^3\,y^3 - 6\,x^2\,y^4 + x^3\,y^4 - 2\,x\,y^5 + 3\,x^2\,y^5 + 3\,x\,y^6 + y^7$$

By Proposition 12, if I is the principal ideal generated by f then $\sqrt{I}$ is generated by $f_{red} = \dfrac{f}{\mathrm{GCD}\left(f, \dfrac{\partial}{\partial x} f, \dfrac{\partial}{\partial y} f\right)}$. We compute this polynomial f_{red}.

2

```
> g := gcd( f, gcd( diff(f,x), diff(f,y) ) );
```

$$g := x^3 + 2\,x^2\,y - x^2\,y^2 + x\,y^2 - 2\,x\,y^3 - y^4$$

```
> f_red := simplify(f/g);
```

$$f\_red := x^2 + x\,y - x\,y^2 - y^3$$

This is the generator for $\sqrt{I}$.

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```
>
```

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```
> restart;
```

Additional exercise on resultants

a)

We enter the generators of I, the polynomials f and g, and compute $R = \text{Res}(f, g, x)$.

```
> f := x^2+2*y^2-3;
  g := x^2+x*y+y^2;
```

$$f := x^2 + 2y^2 - 3$$

$$g := x^2 + xy + y^2$$

```
> R := resultant( f, g, x );
```

$$R := 3y^4 - 9y^2 + 9$$

b)

To determine if $< R > = I \cap Q[y]$, we compute a Groebner basis G for I w.r.t. lex order, $y < x$.

```
> G := Groebner[Basis]( [f,g], plex(x,y) );
```

$$G := [3 - 3y^2 + y^4, -y^3 + x + 2y]$$

By the Elimination Theorem, the first polynomial g in G (in variable y alone) forms a basis for $I \cap Q[y]$. Since g and R are constant multiples of each other ($R = 3g$), they generate the same ideal. That is, $< R > = I \cap Q[y]$.

c)

To compute A, B such that $A f + B g = R$ we use the method described in the proof of Proposition 9 in Section 3.5.

We find $A^{\sim}$, $B^{\sim}$ such that $A^{\sim} f + B^{\sim} g = 1$, where the coefficients of $A^{\sim}$, $B^{\sim}$ are the entries in the solution x of $\text{Syl}(f, g, x)\, x = [0, 0, 0, 1]^T$. We then multiply these by $R = \text{Res}(f, g, x)$ to get the desired A, B. We must also remember that Maple defines the Sylvester matrix as the transpose of how we define it.

```
> with(LinearAlgebra):
  Syl := Transpose(SylvesterMatrix( f, g, x ));
```

$$Syl := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & y & 1 \\ -3 + 2y^2 & 0 & y^2 & y \\ 0 & -3 + 2y^2 & 0 & y^2 \end{bmatrix}$$

```
> C := LinearSolve( Syl, Vector( [0,0,0,1] ) );
```

OK.

$$C := \begin{bmatrix} \dfrac{y}{3(-3y^2+y^4+3)} \\ \dfrac{-3+2y^2}{3(-3y^2+y^4+3)} \\ -\dfrac{y}{3(-3y^2+y^4+3)} \\ -\dfrac{-3+y^2}{3(-3y^2+y^4+3)} \end{bmatrix}$$

Above we have the coefficients for $A^{\sim}$, $B^{\sim}$. We multiply these by R, and then build A, B.

```
> C := simplify(R*C);
```

$$C := \begin{bmatrix} y \\ -3+2y^2 \\ -y \\ 3-y^2 \end{bmatrix}$$

```
> A := C[1]*x+C[2];
  B := C[3]*x+C[4];
```

$$A := x\,y - 3 + 2\,y^2$$
$$B := -x\,y + 3 - y^2$$

This gives us the polynomials $A, B \in Q[x, y]$ such that $A\,f + B\,g = R$. We verify this equation.

```
> expand( A*f+B*g-R );
```

$$0$$

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```
>
```

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Error in my grevlex monomial ordering. Ignore grevlex GB's Calculated.
OK

```
> restart;
```

Additional exercise on Buchberger's algorithm

Fortunately I did all of this work for the last assignment!

Implementation of the Division Algorithm

We program the multivariate division algorithm. Our procedure takes as input a polynomial f to divide, an ordered s-tuple $[f_1, ..., f_s]$ to divide by, a variable ordering X, and a procedure $\mathrm{LT}(g, X)$ that computes the leading term of a polynomial g with respect to a certain monomial order. The output from our procedure is $[[a_1, ..., a_s], r]$ satisfying the conditions given in the division algorithm. The procedure uses a global procedure *monom_divis* that tests whether one monomial term is divisible by another monomial term. If the optional parameter *verbose* = *true* is given, the procedure will print out its intermediate calculations.

```
> DIVIDE := proc( f, fL::list, X::list, LT::procedure )
  local s, a, r, p, t, i, verb;
  global monom_divis;
    verb := false;
    if (nargs > 4) then for t in args[5..-1] do
      if (op(1,t) = verbose) then verb := op(2,t) fi;
    od fi;
    s := nops(fL);
    for i from 1 to s do a[i] := 0 od;
    (p,r) := (f,0);
    while (p <> 0) do
      if (verb) then print('p' = p) fi;
      i := 1;
      while (i <= s) and not(monom_divis( LT(p,X), LT(fL[i],X),
  X )) do i := i+1 od;
      if (i > s) then
        (p,r) := (p-LT(p,X),r+LT(p,X));
        if (verb) then print('r' = r) fi;
      else
        t := LT(p,X)/LT(fL[i],X);
        (p,a[i]) := (p-expand(t*fL[i]),a[i]+t);
        if (verb) then print(`a[`||i||`]` = a[i]) fi;
      fi;
    od;
    return [[seq(a[i], i=1..s)],r];
  end:
```

We write a procedure to compute the multidegree of a polynomial f in variables X. The multidegree is dependent on the monomial ordering chosen, so as in the division algorithm we input a procedure to compute leading terms.

```
> multideg := proc( f, X::list, LT::procedure )
    if type( f, `+` ) then return multideg( LT(f,X), X, LT )
  fi;
    return map2( degree, f, X );
  end:
```

We write a procedure to test if one monomial term m_1 is divisible by another monomial term m_2 in variables X. Note that the property of divisibility is independent of the monomial ordering chosen.

```
> monom_divis := proc( m1, m2, X::list )
  local a, i;
  global multideg;
    if type( m1, `+` ) or type( m2, `+` ) then error "inputs
  should be monomials" fi;
    a := multideg( m1, X ) - multideg( m2, X );
    for i in a do if (i < 0) then return false fi od;
    return true;
  end:
```

We write a procedure to compute leading terms with respect to lexicographic order.

```
> LTlex := proc( f, X::list ) local c, m;
    c := lcoeff( f, X, 'm' );
    return c*m;
  end:
```

We write a procedure to compute leading terms with respect to graded lexicographic order.

```
> LTgrlex := proc( f, X::list ) local d, g, t;
    if not(type( f, `+` )) then return f fi;
    d := max( seq( degree(t), t=f ) );
    g := add( `if`( degree(t) = d, t, 0 ), t=f );
    return LTlex(g,X);
  end:
```

We write a procedure to compute leading terms with respect to graded reverse lexicographic order.

```
> LTgrevlex := proc( f, X::list ) local d, g, t, c, m;
    if not(type( f, `+` )) then return f fi;
    d := max( seq( degree(t), t=f ) );
    g := add( `if`( degree(t) = d, t, 0 ), t=f );
    c := tcoeff( g, X, 'm' );
    return c*m;
  end:
```

Implementation of Buchberger's algorithm

We begin by programming a procedure to calculate S-polynomials. This procedure computes

S(f, g) with respect to the monomial order defined by the variable ordering X, and the leading-term procedure LT(g, X). If the optional parameter *fractionfree* = *true* is given then the resulting S-polynomial will be scaled so that none of its coefficients are fractions.

```
> S_poly := proc( f, g, X::list, LT )
  local ltf, ltg, a, b, c, lcm, i, fractfree;
    fractfree := false;
    if (nargs > 4) then for c in args[5..-1] do
      if (op(1,c) = fractionfree) then fractfree := op(2,c)
  fi;
    od fi;
    (ltf,ltg) := (LT(f,X),LT(g,X));
    (a,b) := multideg( ltf, X ),multideg( ltg, X );
    lcm := mul( X[i]^max(a[i],b[i]), i=1..nops(a) );
    if (fractfree) then
      c := ilcm( lcoeff( ltf, X ), lcoeff( ltg, X ) );
      lcm := c*lcm;
    fi;
    return expand(lcm/ltf*f)-expand(lcm/ltg*g);
  end:
```

We program Buchberger's algorithm as given in lecture and in the textbook. Our procedure takes as input a list of generators $[f_1, ..., f_s]$ for an ideal, a variable ordering X, and a procedure LT(g, X) that computes leading terms with respect to a certain monomial order. It outputs a Groebner basis $[g_1, ..., g_t]$ for the ideal. This Groebner basis will contain no new duplicate polynomials and will preserve the order of the original polynomials. The procedure uses global procedures *S_poly* for computing S-polynomials and *DIVIDE* to perform the multivariate division algorithm. The optional parameters *verbose* and *fractionfree* can be specified true or false to determine if intermediate calculations will be shown and if S-polynomials will be scaled to be fraction free.

```
> Buchberger := proc( F::list, X::list, LT::procedure )
  local G, G_set, f, g, S, r, i, j, k, verb, fractfree;
  global DIVIDE, S_poly;
    (verb,fractfree) := (false,false);
    if (nargs > 3) then for r in args[4..-1] do
      if (op(1,r) = verbose) then verb := op(2,r);
      elif (op(1,r) = fractionfree) then fractfree := op(2,r)
  fi;
    od fi;
    (G,G_set) := (F,{op(F)});
    j := 1;
    while (j <= nops(G)) do
      g := G[j];
      k := nops(G);
```

```
        i := 1;
        while (i < j) do
          f := G[i];
          S := S_poly( f, g, X, LT, fractionfree=fractfree );
          r := DIVIDE( S, G[1..k], X, LT );
          if (verb) then print('S'(f,g) = S, div = r, 'G' = k)
fi;
          r := r[-1];
          if (r <> 0) and not(member( r, G_set )) then
            (G,G_set) := ([op(G),r],G_set union {r}) fi;
          i := i+1;
        od;
        j := j+1;
      od;
      return G;
    end:
```

We write a procedure to take any Groebner basis for an ideal with respect to a given monomial order and transform it into the unique reduced Groebner basis for this ideal with respect to the monomial order.

```
> GroebnerReduce := proc( GB::list, X::list, LT::procedure )
  local G, rG, g, r, verb;
  global DIVIDE;
    verb := false;
    if (nargs > 3) then for r in args[4..-1] do
      if (op(1,r) = verbose) then verb := op(2,r) fi;
    od fi;
    (rG,G) := ([],GB);
    while (G <> []) do
      (g,G) := (G[1],G[2..-1]);
      r := DIVIDE( g, [op(rG),op(G)], X, LT );
      if (verb) then print('g' = g, rem = r, 'G' =
  nops(rG)+nops(G)) fi;
      r := r[-1];
      if (r <> 0) then rG := [op(rG),r/lcoeff( LT(r,X), X )]
  fi;
    od;
    return rG;
  end:
```

Problem 1

We enter the given polynomials and the variable ordering.

```
> X := [x,y,z];
  f1 := x*y-1;
```

```
f2 := x*z-1;
```

$$X := [x, y, z]$$
$$f1 := x\,y - 1$$
$$f2 := x\,z - 1$$

We compute the Groebner basis with respect to lex order, grlex order *and* grevlex order (with the given variable ordering), for the ideal generated by the given polynomials.

```
> G_lex := Buchberger( [f1,f2], X, LTlex, verbose=true );
  G_grlex := Buchberger( [f1,f2], X, LTgrlex, verbose=true );
  G_grevlex := Buchberger( [f1,f2], X, LTgrevlex, verbose=true
  );
```

$$\mathrm{S}(x\,y - 1, x\,z - 1) = -z + y,\ div = [[0, 0], -z + y],\ G = 2$$
$$\mathrm{S}(x\,y - 1, -z + y) = x\,z - 1,\ div = [[0, 1, 0], 0],\ G = 3$$
$$\mathrm{S}(x\,z - 1, -z + y) = -y + x\,z^2,\ div = [[0, z, \text{-}1], 0],\ G = 3$$
$$G\_lex := [x\,y - 1, x\,z - 1, -z + y]$$

Already done

$$\mathrm{S}(x\,y - 1, x\,z - 1) = -z + y,\ div = [[0, 0], -z + y],\ G = 2$$ OK
$$\mathrm{S}(x\,y - 1, -z + y) = x\,z - 1,\ div = [[0, 1, 0], 0],\ G = 3$$
$$\mathrm{S}(x\,z - 1, -z + y) = -y + x\,z^2,\ div = [[0, z, \text{-}1], 0],\ G = 3$$
$$G\_grlex := [x\,y - 1, x\,z - 1, -z + y]$$
$$\mathrm{S}(x\,y - 1, x\,z - 1) = -z + y,\ div = [[0, 0], -z + y],\ G = 2$$
$$\mathrm{S}(x\,y - 1, -z + y) = -z + x\,y^2,\ div = [[y, 0, 1], 0],\ G = 3$$
$$\mathrm{S}(x\,z - 1, -z + y) = x\,y - 1,\ div = [[1, 0, 0], 0],\ G = 3$$
$$G\_grevlex := [x\,y - 1, x\,z - 1, -z + y]$$

Now we reduce each Groebner basis and print it with its set of leading terms.

```
> G_lex := GroebnerReduce( G_lex, X, LTlex ):
  print( lex, G = G_lex, LT(G) = map( LTlex, G_lex, X ) );
  G_grlex := GroebnerReduce( G_grlex, X, LTgrlex ):
  print( grlex, G = G_grlex, LT(G) = map( LTgrlex, G_grlex, X )
  );
  G_grevlex := GroebnerReduce( G_grevlex, X, LTgrevlex ):
  print( grevlex, G = G_grevlex, LT(G) = map( LTgrevlex,
  G_grevlex, X ) );
```

$$lex,\ G = [x\,z - 1, -z + y],\ \mathrm{LT}(G) = [x\,z, y]$$
$$grlex,\ G = [x\,z - 1, -z + y],\ \mathrm{LT}(G) = [x\,z, y]$$
$$grevlex,\ G = [x\,y - 1, -y + z],\ \mathrm{LT}(G) = [x\,y, z]$$

Problem 2

We enter the given polynomials and the variable ordering.

```
> X := [w,x,y,z];
  f1 := 3*x-6*y-2*z;
```

```
f2  :=  2*x-4*y+4*w;
f3  :=  x-2*y-z-w;
```

$$X := [w, x, y, z]$$

$$f1 := 3\,x - 6\,y - 2\,z$$

$$f2 := 2\,x - 4\,y + 4\,w$$

$$f3 := x - 2\,y - z - w$$

We compute the Groebner basis with respect to lex order, grlex order *and* grevlex order (with the given variable ordering), for the ideal generated by the given polynomials.

```
> G_lex := Buchberger( [f1,f2,f3], X, LTlex );
  G_grlex := Buchberger( [f1,f2,f3], X, LTgrlex );
  G_grevlex := Buchberger( [f1,f2,f3], X, LTgrevlex );
```

$$G\_lex := [3\,x - 6\,y - 2\,z, 2\,x - 4\,y + 4\,w, x - 2\,y - z - w]$$

$$G\_grlex := [3\,x - 6\,y - 2\,z, 2\,x - 4\,y + 4\,w, x - 2\,y - z - w]$$

$$G\_grevlex := [3\,x - 6\,y - 2\,z, 2\,x - 4\,y + 4\,w, x - 2\,y - z - w]$$

Now we reduce each Groebner basis and print it with its set of leading terms.

```
> G_lex := GroebnerReduce( G_lex, X, LTlex ):
  print( lex, G = G_lex, LT(G) = map( LTlex, G_lex, X ) );
  G_grlex := GroebnerReduce( G_grlex, X, LTgrlex ):
  print( grlex, G = G_grlex, LT(G) = map( LTgrlex, G_grlex, X )
  );
  G_grevlex := GroebnerReduce( G_grevlex, X, LTgrevlex ):
  print( grevlex, G = G_grevlex, LT(G) = map( LTgrevlex,
  G_grevlex, X ) );
```

$$lex, G = \left[x - 2\,y - \frac{2\,z}{3}, w + \frac{z}{3} \right], \mathrm{LT}(G) = [x, w] \quad \checkmark$$

$$grlex, G = \left[x - 2\,y - \frac{2\,z}{3}, w + \frac{z}{3} \right], \mathrm{LT}(G) = [x, w] \quad \checkmark$$

$$grevlex, G = \left[y - \frac{x}{2} \cancel{- w, z + 3\,w} \right] \cancel{, \mathrm{LT}(G) = [y, z]}$$

Problem 3

We enter the given polynomials and the variable ordering.

```
> X := [x,y,z];
  f1 := x^2+y+z-1;
  f2 := x+y^2+z-1;
  f3 := x+y+z^2-1;
```

$$X := [x, y, z]$$

$$f1 := x^2 + y + z - 1$$

$$f2 := x + y^2 + z - 1$$

$$f3 := x + y + z^2 - 1$$

We compute the Groebner basis with respect to lex order, grlex order *and* grevlex order (with the given variable ordering), for the ideal generated by the given polynomials.

```
> G_lex := Buchberger( [f1,f2,f3], X, LTlex );
  G_grlex := Buchberger( [f1,f2,f3], X, LTgrlex );
  G_grevlex := Buchberger( [f1,f2,f3], X, LTgrevlex );
```

$$G\_lex := \left[x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1, y + y^4 + 2y^2 z - 2y^2 + z^2 - z,\right.$$
$$y^3 + zy + z^2 y^2 + z^3 - z^2 - y^2, y^2 + z - y - z^2, -z^4 + z^2 - 2z^2 y, 2z^4 - 2z^3 - \frac{1}{2}z^6 + \frac{1}{2}z^2,$$
$$\left.-2z^4 + 2z^3 + \frac{1}{2}z^6 - \frac{1}{2}z^2\right]$$

$$G\_grlex := [x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1]$$

$$G\_grevlex := [x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1]$$

Now we reduce each Groebner basis and print it with its set of leading terms.

```
> G_lex := GroebnerReduce( G_lex, X, LTlex ):
  print( lex, G = G_lex, LT(G) = map( LTlex, G_lex, X ) );
  G_grlex := GroebnerReduce( G_grlex, X, LTgrlex ):
  print( grlex, G = G_grlex, LT(G) = map( LTgrlex, G_grlex, X )
  );
  G_grevlex := GroebnerReduce( G_grevlex, X, LTgrevlex ):
  print( grevlex, G = G_grevlex, LT(G) = map( LTgrevlex,
  G_grevlex, X ) );
```

$$lex, G = \left[x + y + z^2 - 1, y^2 + z - y - z^2, \frac{1}{2}z^4 - \frac{1}{2}z^2 + z^2 y, z^6 - 4z^4 + 4z^3 - z^2\right],$$

$$LT(G) = [x, y^2, z^2 y, z^6]$$

$$grlex, G = [x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1], LT(G) = [x^2, y^2, z^2]$$

$$\cancel{grevlex, G = [x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1], LT(G) = [x^2, y^2, z^2]}$$

Problem 4

We enter the given polynomials and the correct variable ordering, **not the one given in the assignment description!**

```
> X := [x,y,z];
  f1 := x-z^4;
  f2 := y-z^5;
```

$$X := [x, y, z]$$
$$f1 := x - z^4$$
$$f2 := y - z^5$$

We compute the Groebner basis with respect to lex order, grlex order *and* grevlex order (with the given variable ordering), for the ideal generated by the given polynomials.

```
> G_lex := Buchberger( [f1,f2], X, LTlex );
  G_grlex := Buchberger( [f1,f2], X, LTgrlex );
  G_grevlex := Buchberger( [f1,f2], X, LTgrevlex );
```

$$G\_lex := [x - z^4, y - z^5]$$ ✓

$$G\_grlex := [x - z^4, y - z^5, -z\,x + y, -x^2 + z^3\,y, -z^2\,y^2 + x^3, -y^3\,z + x^4]$$

$$G\_grevlex := [x - z^4, y - z^5, -z\,x + y, -x^2 + z^3\,y, -z^2\,y^2 + x^3, -y^3\,z + x^4, -y^4 + x^5]$$

Now we reduce each Groebner basis and print it with its set of leading terms.

```
> G_lex := GroebnerReduce( G_lex, X, LTlex ):
  print( lex, G = G_lex, LT(G) = map( LTlex, G_lex, X ) );
  G_grlex := GroebnerReduce( G_grlex, X, LTgrlex ):
  print( grlex, G = G_grlex, LT(G) = map( LTgrlex, G_grlex, X )
  );
  G_grevlex := GroebnerReduce( G_grevlex, X, LTgrevlex ):
  print( grevlex, G = G_grevlex, LT(G) = map( LTgrevlex,
  G_grevlex, X ) );
```

$$lex, G = [x - z^4, y - z^5], \mathrm{LT}(G) = [x, y]$$ ✓

$$grlex, G = [-x + z^4, z\,x - y, -x^2 + z^3\,y, z^2\,y^2 - x^3, -y^3\,z + x^4],$$
$$\mathrm{LT}(G) = [z^4, z\,x, z^3\,y, z^2\,y^2, x^4]$$ ✓

$$grevlex, G = [-x + z^4, z\,x - y, -x^2 + z^3\,y, z^2\,y^2 - x^3, y^3\,z - x^4, -y^4 + x^5],$$
$$\mathrm{LT}(G) = [z^4, z\,x, z^3\,y, z^2\,y^2, y^3\,z, x^5]$$

Problem 5

We enter the given polynomials and the variable ordering.

```
> X := [t,x,y,z];
  f1 := t^2+x^2+y^2+z^2;
  f2 := t^2+2*x^2-x*y-z^2;
  f3 := t+y^3-z^3;
```

$$X := [t, x, y, z]$$
$$f1 := t^2 + x^2 + y^2 + z^2$$
$$f2 := t^2 + 2\,x^2 - x\,y - z^2$$
$$f3 := t + y^3 - z^3$$

We compute the Groebner basis with respect to lex order, grlex order *and* grevlex order (with the given variable ordering), for the ideal generated by the given polynomials.

```
> G_lex := Buchberger( [f1,f2,f3], X, LTlex );
  G_grlex := Buchberger( [f1,f2,f3], X, LTgrlex );
  G_grevlex := Buchberger( [f1,f2,f3], X, LTgrevlex );
```

$$G\_lex := [t^2 + x^2 + y^2 + z^2, t^2 + 2\,x^2 - x\,y - z^2, t + y^3 - z^3, -x^2 + y^2 + 2\,z^2 + x\,y,$$
$$2\,y^2 + 3\,z^2 + y^6 - 2\,y^3\,z^3 + z^6 + x\,y, -5\,y^4 - 13\,z^2\,y^2 - 5\,z^6\,y^2 - 9\,z^4 - y^{12} + 4\,y^9\,z^3 - 6\,y^6\,z^6$$

$+ 4 y^3 z^9 - 5 y^8 + 10 z^3 y^5 - z^{12} - 6 z^2 y^6 + 12 z^5 y^3 - 6 z^8, 5 y^3 + 7 y z^2 - 3 x z^2 - x z^6 + 5 y^7$
$+ 3 y^5 z^2 + y^{11} - 4 y^8 z^3 + 5 y^5 z^6 - 10 y^4 z^3 - 6 y^2 z^5 - 2 y^2 z^9 + 3 z^6 y]$

$G\_grlex := [t^2 + x^2 + y^2 + z^2, t^2 + 2 x^2 - x y - z^2, t + y^3 - z^3, -x^2 + y^2 + 2 z^2 + x y]$

$$G\_grevlex := \Big[t^2 + x^2 + y^2 + z^2, t^2 + 2 x^2 - x y - z^2, t + y^3 - z^3, 2 t^2 + 3 x^2 + y^2 - x y,$$
$$z x y - 2 z x^2 - 2 y x^2 - 3 x^3 - z t^2 - 2 y t^2 - 2 x t^2 + t,$$
$$5 z x^4 + 7 y x^4 - 3 x^5 + 5 z x^2 t^2 + 8 y x^2 t^2 - 5 x^3 t^2 + z t^4 + 2 y t^4 - 2 t^4 x - x y t - x^2 t - t^3,$$
$$x^3 y + 7 z x^3 + 7 x^4 + 2 z y t^2 + 4 z x t^2 + 2 t^2 x y + 6 t^2 x^2 + t^4 - z t - 2 x t,$$
$$5 z x^3 + 7 x^3 y - 3 x^4 + z y t^2 + 3 z x t^2 + 6 t^2 x y - 8 t^2 x^2 - 4 t^4 - y t - x t, \frac{32 x^5 y}{25} - \frac{188 x^6}{25}$$
$$- \frac{32 x^3 y t^2}{25} - \frac{57 x^4 t^2}{5} + \frac{z t^4 x}{25} - \frac{43 x y t^4}{25} - \frac{107 t^4 x^2}{25} + \frac{3 z x^2 t}{5} - \frac{x^2 y t}{25} + \frac{39 x^3 t}{25} + \frac{z t^3}{5}$$
$$+ \frac{2 y t^3}{5} + \frac{29 x t^3}{25} - \frac{t^2}{5},$$
$$- \frac{44 x^3 y t^2}{35} + \frac{8 x^4 t^2}{5} + \frac{3 y z t^4}{35} - \frac{z t^4 x}{35} - \frac{32 x y t^4}{35} + \frac{86 t^4 x^2}{35} + \frac{33 t^6}{35} - \frac{z t^3}{7} + \frac{y t^3}{5} - \frac{3 x t^3}{35}$$
$$, \frac{88 y x^4}{35} - \frac{16 x^5}{5} - \frac{2 z x^2 t^2}{7} + \frac{52 y x^2 t^2}{35} - \frac{38 x^3 t^2}{7} - \frac{6 z t^4}{35} - \frac{12 y t^4}{35} - \frac{78 t^4 x}{35} + \frac{2 z x t}{7}$$
$$- \frac{2 x y t}{5} + \frac{6 x^2 t}{35} + \frac{6 t^3}{35}, \frac{243 x^6}{14} + \frac{1881 x^4 t^2}{56} + \frac{15 y z t^4}{28} - \frac{15 z t^4 x}{56} + \frac{15 x y t^4}{56}$$
$$+ \frac{1203 t^4 x^2}{56} + \frac{129 t^6}{28} - \frac{75 z x^2 t}{56} - \frac{27 x^2 y t}{56} - \frac{243 x^3 t}{56} - \frac{75 z t^3}{56} - \frac{3 y t^3}{14} - \frac{207 x t^3}{56}$$
$$+ \frac{33 t^2}{56}, - \frac{162 x^6}{7} - \frac{627 x^4 t^2}{14} - \frac{5 y z t^4}{7} + \frac{5 z t^4 x}{14} - \frac{5 x y t^4}{14} - \frac{401 t^4 x^2}{14} - \frac{43 t^6}{7}$$
$$+ \frac{25 z x^2 t}{14} + \frac{9 x^2 y t}{14} + \frac{81 x^3 t}{14} + \frac{25 z t^3}{14} + \frac{2 y t^3}{7} + \frac{69 x t^3}{14} - \frac{11 t^2}{14}, - \frac{12 y x^4}{35} - \frac{132 x^5}{35}$$
$$+ \frac{z x^2 t^2}{7} - \frac{58 y x^2 t^2}{35} - \frac{38 x^3 t^2}{7} + \frac{4 z t^4}{35} - \frac{37 y t^4}{35} - \frac{68 t^4 x}{35} + \frac{y z t}{7} - \frac{4 x y t}{35} + \frac{3 x^2 t}{5}$$
$$+ \frac{16 t^3}{35}, \frac{44 y x^4}{35} - \frac{8 x^5}{5} - \frac{z x^2 t^2}{7} + \frac{26 y x^2 t^2}{35} - \frac{19 x^3 t^2}{7} - \frac{3 z t^4}{35} - \frac{6 y t^4}{35} - \frac{39 t^4 x}{35} + \frac{z x t}{7}$$
$$- \frac{x y t}{5} + \frac{3 x^2 t}{35} + \frac{3 t^3}{35}, - \frac{324 x^5}{55} + \frac{8 z x^2 t^2}{55} - \frac{112 y x^2 t^2}{55} - \frac{95 x^3 t^2}{11} + \frac{7 z t^4}{55} - \frac{17 y t^4}{11}$$
$$- \frac{173 t^4 x}{55} + \frac{y z t}{5} + \frac{3 z x t}{55} - \frac{13 x y t}{55} + \frac{48 x^2 t}{55} + \frac{37 t^3}{55},$$

$$-\frac{44x^3y}{35}+\frac{8x^4}{5}+\frac{3zyt^2}{35}-\frac{zxt^2}{35}-\frac{32t^2xy}{35}+\frac{86t^2x^2}{35}+\frac{33t^4}{35}-\frac{zt}{7}-\frac{3xt}{35}+\frac{yt}{5}\Bigg]$$

Now we reduce each Groebner basis and print it with its set of leading terms.

```
> G_lex := GroebnerReduce( G_lex, X, LTlex ):
  print( lex, G = G_lex, LT(G) = map( LTlex, G_lex, X ) );
  G_grlex := GroebnerReduce( G_grlex, X, LTgrlex ):
  print( grlex, G = G_grlex, LT(G) = map( LTgrlex, G_grlex, X )
  );
  G_grevlex := GroebnerReduce( G_grevlex, X, LTgrevlex ):
  print( grevlex, G = G_grevlex, LT(G) = map( LTgrevlex,
  G_grevlex, X ) );
```

$$lex, G=[t+y^3-z^3, x^2+y^6-2y^3z^3+y^2+z^6+z^2, 2y^2+3z^2+y^6-2y^3z^3+z^6+xy, 5y^4 + 13z^2y^2+5z^6y^2+9z^4+y^{12}-4y^9z^3+6y^6z^6-4y^3z^9+5y^8-10z^3y^5+z^{12}+6z^2y^6 - 12z^5y^3+6z^8, -5y^3-7yz^2+3xz^2+xz^6-5y^7-3y^5z^2-y^{11}+4y^8z^3-5y^5z^6 + 10y^4z^3+6y^2z^5+2y^2z^9-3z^6y], LT(G)=[t, x^2, xy, y^{12}, xz^6]$$

$$grlex, G=[t^2+xy+2y^2+3z^2, t+y^3-z^3, x^2-y^2-2z^2-xy], LT(G)=[t^2, y^3, x^2]$$

$$grevlex, G=\Bigg[-t^2-2x^2+xy+z^2, 2t^2+3x^2+y^2-xy,$$

$$zxy-2zx^2-2yx^2-3x^3-zt^2-2yt^2-2xt^2+t,$$

$$zx^3+\frac{13x^4}{11}+\frac{13zyt^2}{44}+\frac{25zxt^2}{44}+\frac{2t^2xy}{11}+\frac{25t^2x^2}{22}+\frac{t^4}{4}-\frac{7zt}{44}+\frac{yt}{44}-\frac{13xt}{44}, x^5$$

$$-\frac{2zx^2t^2}{81}+\frac{28yx^2t^2}{81}+\frac{475x^3t^2}{324}-\frac{7zt^4}{324}+\frac{85yt^4}{324}+\frac{173t^4x}{324}-\frac{11yzt}{324}-\frac{zxt}{108}$$

$$+\frac{13xyt}{324}-\frac{4x^2t}{27}-\frac{37t^3}{324},$$

$$x^3y-\frac{14x^4}{11}-\frac{3zyt^2}{44}+\frac{zxt^2}{44}+\frac{8t^2xy}{11}-\frac{43t^2x^2}{22}-\frac{3t^4}{4}+\frac{5zt}{44}+\frac{3xt}{44}-\frac{7yt}{44}\Bigg],$$

$$LT(G)=[z^2, y^2, zxy, zx^3, x^5, x^3y]$$

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```
>
```