Multiplication using a Heap

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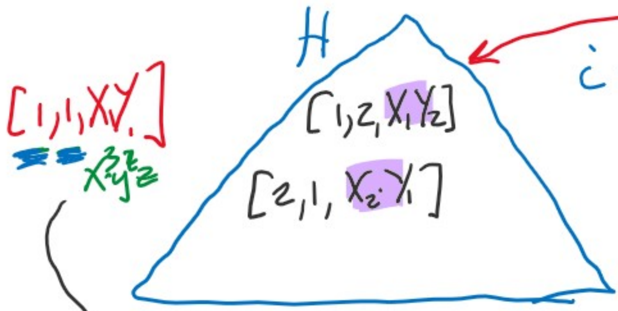
Johnson 1974.

Let $f = f_1 + f_2 + \cdots + f_m = a_1 \cdot X_1 + \cdots + a_m X_m$ s.t. $X_1 > X_2 > \cdots > X_m$

$g = g_1 + g_2 + \cdots + g_\ell = b_1 Y_1 + \cdots + b_\ell Y_\ell$ s.t. $Y_1 > Y_2 > \cdots > Y_\ell$.

Let $h = f \times h$.



$h = a_1 b_1 X_1 Y_1 +$

$j=1$

$h =$ $\quad X_1 \cdot Y_1 > X_1 Y_2 > \cdots > X_1 \cdot Y_\ell$

$i \longrightarrow f_1 g = f_1 g_1 + f_1 g_2 + \cdots + f_1 g_\ell$

$+ f_2 g = f_2 g_1 + f_2 g_2 + \cdots + f_2 g_\ell.$

$+ f_3 g = f_3 g_1 + f_3 g_2 + \cdots + f_3 g_\ell$

$\vdots$

$+ f_m g = f_m g_1 + f_m g_2 + \cdots + f_m g_\ell.$

Initialize H to be an empty heap.

Insert $[1, 1, X_1 Y_1]$ into H.

while $H \neq \emptyset$ do

- extract $H_1 = [i, j, Z]$ from H.
- Set $C := a_i \cdot b_j$. Set $h := 0$;
- if $j=1$ and $i<m$ then insert $[i+1, 1, X_{i+1} \cdot Y_j]$ into H.
- if $j<\ell$ insert $[i, j+1, X_i Y_{j+1}]$ into H.
- while $H \neq \emptyset$ and $H_1 = [\,-\,-\,Z]$ do

    extract $[i, j, Z]$ from H.

    $C \leftarrow C + a_i \cdot b_j$.

    if $j=1$ and $i<m$ then insert $[i+1, 1, X_{i+1} Y_j]$ into H

    if $j<\ell$ then insert $[i, j+1, X_i \cdot Y_{j+1}]$ into H.

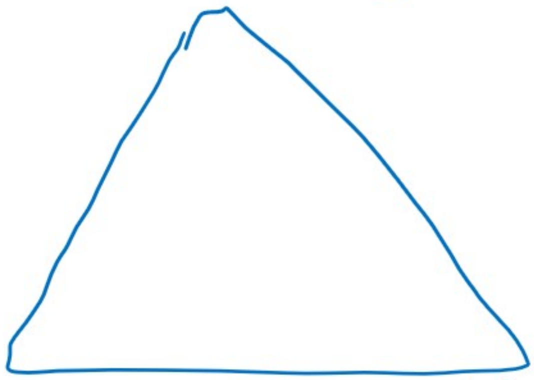
  end.

  if $C \neq 0$ append $[C, Z]$ to h // $h = h + C \cdot Z$.

end.

Example. $f = 2x^2y^2 + 3xy^2 + 4y^3$ in lex. $x>y$

$g = 3x^2 + 5xy$.



$f_1g_1 \quad f_1g_2$

$$f_1 \cdot g = \cdot x^4y^2 + \cdot x^3y^3$$
$$+ f_2 g = \cdot x^3y^2 + \cdot x^2y^3$$
$$+ f_3 \cdot g = \cdot x^2y^3 + \cdot xy^4$$

$$h = 6 \cdot x^4y^2 + 2 \cdot 5 \cdot x^3y^3 + 3 \cdot 3 \cdot x^3y^2 + (3 \cdot 5 \cdot x^2y^3 + 4 \cdot 3 \cdot x^2y^3) + 4 \cdot 5 \cdot xy^4$$

How many monomial comparisons does it do in the worst case?

Every term $f_i g_j$ is inserted into H eventually, once, then eventually it is extracted once from H.

The # insertions is $\#f \cdot \#g = m \cdot \ell$.

$$\begin{aligned}
\#\text{ comparisons} &\leq \#f \cdot \#g \cdot \text{Cost(insertion)} \\
&\quad + \#f \cdot \#g \cdot \text{Cost(extraction)} \\
&\leq \#f \cdot \#g \, (O(\log(\text{max height } H)) \\
&\qquad + O(\log(\text{max height } H))) \\
&= \#f \cdot \#g \cdot O(\log \#f) \\
&= O(m\ell \log m). \quad \text{was } O(m^2\ell).
\end{aligned}$$

If $\underset{m}{\#f} > \underset{\ell}{\#g}$ then $f \cdot g = g \cdot f$ so the cost is

$$O(m\ell \log \min(m,\ell)).$$