Simplifying Bases

October 31, 2023 10:25 AM

How can we simplify a basis B for an ideal I in $k[x_1, \dots, x_n]$?

Lemma (very useful Lemma)

Let I be an ideal in $k[x_1, \dots, x_n]$ with basis $\{f_1, \dots, f_s\}$.

(i) If $s \in k \setminus \{0\}$ then $B \setminus \{f_i\} \cup \{s f_i\}$ is a basis for I.

(ii) If $h \in k[x_1, \dots, x_n]$ then $B \setminus \{f_j\} \cup \{g_j\}$ is a basis for I where $g_j = f_j - h \cdot f_i$.

Proof.

$$I = \langle f_1, \dots, f_i, \dots, f_j, \dots, f_s \rangle$$
$$J = \langle f_1, \dots, f_i, \dots, g_j, \dots, f_s \rangle$$

$(f_j \in J)$ $\quad f_j = 1 \cdot \underset{\in J}{g_j} + h \cdot \underset{\in J}{f_i} \in J.$

$(g_j \in I)$ $\quad g_j = 1 \cdot \underset{\in I}{f_j} - h \cdot \underset{\in I}{f_i} \in I.$

Example. Let $\langle \overset{f_1}{x^2+y^2-1}, \overset{f_2}{x+y} \rangle = I$

Let $f_3 = f_1 - x \cdot f_2 = x^2+y^2-1-(x^2+xy) = -xy+y^2-1.$

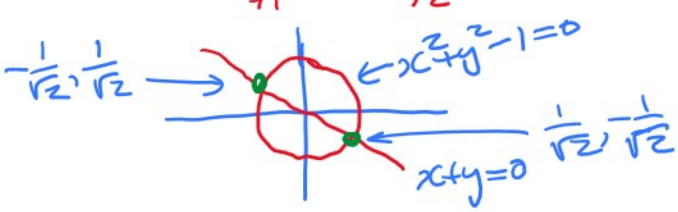
$\Rightarrow I = \langle \overset{f_2}{x+y}, \overset{f_3}{-xy+y^2-1} \rangle$ by VUL (ii).

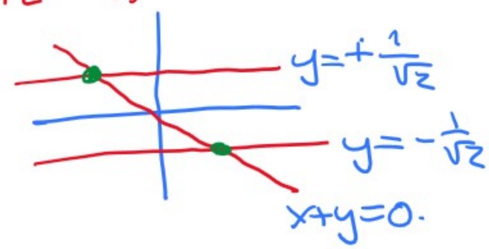
Let $f_4 = f_3 + y f_2 = \cancel{-xy} + y^2 - 1 + \cancel{xy} + y^2 = 2y^2 - 1.$

$\Rightarrow I = \langle \underset{f_2}{x+y}, \underset{f_4}{2y^2-1} \rangle$ by VUL (ii). $\quad f_5 = \frac{1}{2} f_4.$

$I = \langle x+y, \underset{f_5}{y^2 - \frac{1}{2}} \rangle$ by VUL (i).

Notice that $\mathbb{V}(\underset{f_1}{x^2+y^2-1}, \underset{f_2}{x+y}) = \mathbb{V}(\underset{f_2}{x+y}, \underset{f_5}{y^2-\frac{1}{2}})$. $\quad y^2 = \frac{1}{2}$, $y = \pm \frac{1}{\sqrt{2}}$





What is the connection between ideals in $k[x_1,\dots,x_n]$ and varieties in k^n?

Prop 4. If $\langle f_1,\dots,f_s\rangle \overset{\text{VUL}}{=} \langle g_1,\dots,g_t\rangle$ then $\mathbb{V}(f_1,\dots,f_s)=\mathbb{V}(g_1,\dots,g_t)$.

The converse is not true in general. It's true if k is algebraically closed e.g. $k=\mathbb{C}$.

Ex. In $\mathbb{R}^2$ $\mathbb{V}(x^2+y^2+1)=\emptyset=\mathbb{V}(1)$
but $\langle x^2+y^2+1\rangle \neq \langle 1\rangle = \mathbb{R}[x,y]$.

Questions about ideals in $k[x_1,\dots,x_n]$.
Let I be an ideal in $k[x_1,\dots,x_n]$.
Does I have a finite basis? Yes. Hilbert basis Theorem.
What is a good basis for I? Grobner basis.
How can I test if $f\in I$? Divide by a Grobner basis for I.

Def. If $I=\underline{\langle g\rangle}$ for some non-zero $g\in k[x_1,\dots,x_n]$ then I is called a principle ideal. $G=\{g\}$ is a G.B. for I.
$I=\{h\cdot g : h\in k[x_1,\dots,x_n]\}$ so $f\in I \Leftrightarrow g | f$.

CLO <u>1.5 Ideals in $k[x]$.</u>

Theorem Let I be an ideal in $k[x]$.
Then $I=\langle f\rangle$ for some $f\in k[x]$.

Proof. Case $I=\underline{\{0\}}$. Take $f=0$. $I=\langle 0\rangle=\{0\}$..
Case $I\neq\{0\}$ Let f be a non-zero polynomial in I of least degree. I claim $I=\langle f\rangle$.

Proof. $\langle f\rangle \subset I$: Let $g\in\langle f\rangle=\{h\cdot f\}\Rightarrow g=h\cdot f$ for some h.
But $f\in I\Rightarrow g\in I$.

Let $g \in I$. Is g a multiple of f?

$I \subset \langle f \rangle$: Consider $g \div f$. $\underline{k[x]}$

$\exists\, q, r \in k[x]$ s.t. $g = qf + r$ with $r = 0$ or $\deg(r) < \deg(f)$.

$\Rightarrow r = \underset{I}{\underset{\cap}{g}} - \underset{I}{\underset{\cap}{qf}} \Rightarrow r \in I$.

If $r \neq 0 \Rightarrow \underset{r \in I.}{\deg(r) < \deg(f)} \Rightarrow \boxed{\times}$ contradicts our choice of f

$\Rightarrow r = 0 \Rightarrow g = f \cdot q \Rightarrow g \in \langle f \rangle$.

How do we find f ??

Lemma. Let $I = \langle f_1, \dots, f_s \rangle \subset k[x]$.
$I = \langle g \rangle$ where $g = \gcd(f_1, \dots, f_s)$.

Proof $\langle g \rangle \subset I$ For $I = \langle f_1, f_2, f_3 \rangle$.

$\exists\, s_1, s_2$ s.t. $s_1 f_1 + s_2 f_2 = \gcd(f_1, f_2) = h$. by the EEA.

$\exists\, t_1, t_2$ s.t. $t_1 h + t_2 f_3 = \gcd(h, f_3) = \gcd(f_1, f_2, f_3) = g$.

$\Rightarrow g = t_1(s_1 f_1 + s_2 f_2)h + t_2 f_3$

$= \underbrace{t_1 h s_1}_{\in R} f_1 + \underbrace{t_1 h s_2}_{\in R} f_2 + \underbrace{t_2}_{\in R} f_3 \in I$.

Exercise. Let $V = \mathbb{V}(x^2 + y^2 - 3, x^2 - y^2 - 1)$

Solve V by simplifying the basis for

$I = \langle \overset{f_1}{x^2 + y^2 - 3}, \overset{f_2}{x^2 - y^2 - 1} \rangle$ using the VUL.

Let $f_3 = f_1 - f_2 = (y^2 - 3) - (-y^2 - 1) = 2y^2 - 2$.

$\Rightarrow I = \langle \underset{f_3}{\cancel{2}y^2 - \cancel{2}}, \underset{f_2}{x^2 - y^2 - 1} \rangle$ by VUL (ii)

$= \langle \underset{f_4}{y^2 - 1}, \underset{f_2}{x^2 - y^2 - 1} \rangle$ by VUL (i)

Let $f_5 = f_2 + f_4 = x^2 - 2$.

$\Rightarrow I = \langle \overset{f_4}{y^2 - 1}, \overset{f_5}{x^2 - 2} \rangle$.

$$\Rightarrow \quad I = \langle \overset{f_4}{y^2-1},\ \overset{f_5}{x^2-2} \rangle.$$

Hence $V = \mathbb{V}(y^2-1, x^2-2) = \{(\pm\sqrt{2}, \pm 1)\}$.

Note. $\{x^2-2, y^2-1\}$ is a GB for I wrt any monomial ordering. If $\gcd(LT(g_i), LT(g_j)) = 1$ $\forall\, i \neq j$ then $\{g_1, \dots, g_t\}$ is a GB for $I = \langle g_1, \dots, g_t \rangle$.

E.g. $\langle x+y,\ y+z,\ z-1 \rangle = I$.

$\{x+y,\ y+z,\ z-1\}$ for lex with $\underline{x > y > z}$

is a GB for I. $\uparrow$ in row Echelon form.