Monomial Ideals

November 2, 2023 9:20 AM

CLO 2.4

Def An ideal $I \subset k[x_1,\dots,x_n]$ is a <u>monomial ideal</u> if $\exists A \subset \mathbb{Z}^n_{\geq 0}$, possibly infinite, s.t.

$$I = \{ \sum_{\alpha \in A} h_\alpha x^\alpha : h_\alpha \in k[x_1,\dots,x_n] \}.$$

We write $I = \langle x^\alpha : \alpha \in A \rangle$ and we say I is generated by $\{x^\alpha : \alpha \in A\}$.

$f_3 = f_2 - f_1 = y$

Ex. $I = \langle \underset{f_1}{x}, \underset{f_2}{x+y} \rangle = \langle x, y \rangle$ by VUL.

$I = \langle x+1 \rangle$ is not a monomial ideal.

Lemma 2 Let $I = \langle x^\alpha : \alpha \in A \subset \mathbb{Z}^n_{\geq 0} \rangle$.
Then $x^\beta \in I \iff x^\alpha \mid x^\beta$ for some $\alpha \in A$.

Proof. ($\Leftarrow$) $x^\alpha \mid x^\beta \Rightarrow x^\beta = \underset{\in I}{x^\alpha} \cdot \underset{h}{x^\delta} \in I$ by def (iii).

($\Rightarrow$) $x^\beta \in I \Rightarrow x^\beta = \sum_{i=1}^{s} h_i \cdot x^{\alpha(i)}$ for some $\alpha(i) \in A$ and $h_i \in k[x_1,\dots,x_n]$.

We will prove that one of the $x^{\alpha(i)} \mid x^\beta$, by induction on s.

Case $s=1$: We have $x^\beta = h_1 \cdot x^{\alpha(1)}$. $\Rightarrow x^{\alpha(1)} \mid x^\beta$ $\boxtimes$.

CASE $s>1$. We have $x^\beta = \sum h_i x^{\alpha(i)}$

If $x^{\alpha(1)} \mid x^\beta$ we are done.
Otherwise $x^{\alpha(1)} \nmid x^\beta$. Let

$$x^\beta = r_1 \cdot x^{\alpha(1)} + \underbrace{r_2 x^{\alpha(2)} + \dots + r_s x^{\alpha(s)}}_{x^{\alpha(1)} \nmid \text{ any terms has}}.$$

all terms divisible by $x^{\alpha(1)}$.

If $x^{\alpha(1)} \nmid x^\beta$ and $x^{\alpha(1)} \nmid [r_2 x^{\alpha(2)} + \dots + r_3 x^{\alpha(1)}]$ then $r_1 = 0$.

If $x^{\alpha(1)} \nmid x^\beta$ and $x^{\alpha(1)} \nmid [r_2 x^{\alpha(2)} + \cdots + r_s x^{\alpha(s)}]$ then $r_1 = 0$.

$$\Rightarrow x^\beta = r_2 x^{\alpha(2)} + \cdots + r_s x^{\alpha(s)}.$$

By induction on s one of $x^{\alpha(2)}, \dots, x^{\alpha(s)}$ must divide x^β.

Visualizing Monomial Ideals

Let $\alpha + \mathbb{Z}^n_{\geq 0} = \{\alpha + \beta : \beta \in \mathbb{Z}^n_{\geq 0}\}$

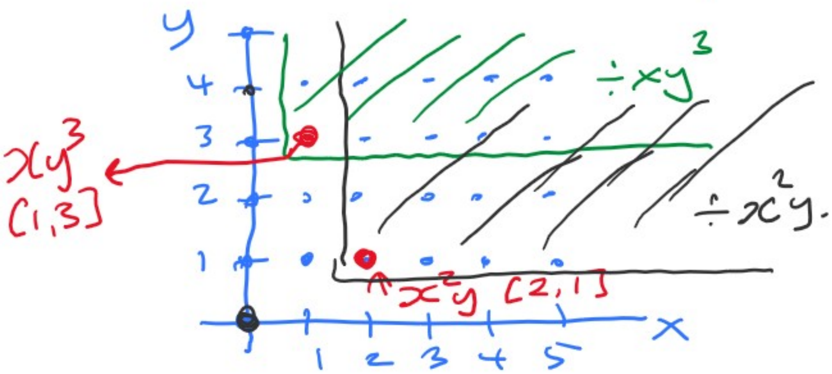
Thus $\{x^\beta : \beta = \alpha + \mathbb{Z}^n_{\geq 0}\}$ is all monomials divisible by x^α.

If $I = \langle x^2y, xy^3 \rangle$ then by Lemma 2

$x^\beta \in I \Rightarrow \underline{x^2y} \mid x^\beta$ or $xy^3 \mid x^\beta$.

$\Rightarrow \beta \in [2,1] + \mathbb{Z}^n_{\geq 0}$ or $\beta \in [1,3] + \mathbb{Z}^n_{\geq 0}$.

Which we can visualize by the shaded area in



Theorem 5 (Dickson's Lemma). $\langle \underline{x_1^3}, x_1^5, x_1^{151}, \dots \rangle$

A monomial ideal $I = \langle x^\alpha : \alpha \in A \subset \mathbb{Z}^n_{\geq 0} \rangle$ for $n \geq 1$ has a finite basis i.e.

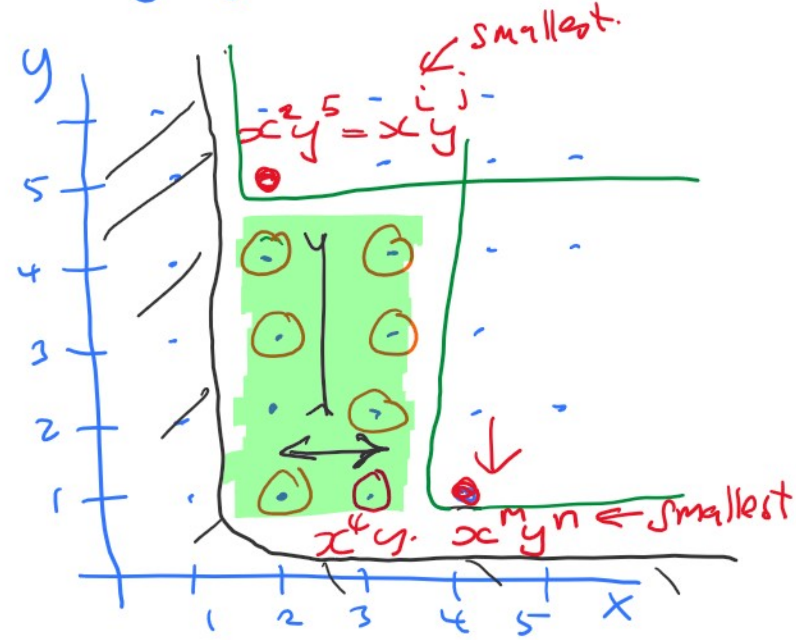
$$I = \langle x^{\alpha(1)}, x^{\alpha(2)}, \dots, x^{\alpha(s)} \rangle \text{ for some } s \in \mathbb{N}.$$

Proof. $(n=1)$ $I \subset k[x_1]$.

In 1.5 $I = \langle g \rangle$ where $g = \gcd(x^\alpha)$. $\square$

smallest.

(n=2). $I \subset k[x,y]$.



Let $x^iy^j \in I$ with i minimal.

Let $x^my^n \in I$ with n minimal.

Let $J = \langle x^iy^j, x^my^n \rangle$.

I is generated by x^iy^j and x^my^n and a subset of at most $(m-i)(j-n)$.

↑

Number of monomials in green rectangle.