Groebner bases

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Example. $I = \langle \overset{f_1}{xy} - 1, \overset{f_2}{xz} - 1 \rangle$

$xy > 1 \quad xz > 1$

We cant use VUL (ii) as $LT(f_1) \nmid LT(f_2)$ and $LT(f_2) \nmid LT(f_1)$.

Consider

$$I \ni z f_1 - y f_2 = \cancel{xyz} - z - \cancel{xyz} + y = y - z \in I$$

We have $I = \langle \overset{f_1}{xy - 1}, \overset{f_2}{xz - 1}, \overset{f_3}{y - z} \rangle$.

Lets use $>_{lex}$ with $y > z$.

$$I \ni f_1 - x f_3 = xy - 1 - xy + xz = \overset{f_2}{xz - 1} \in I.$$

By VUL (i) $I = \langle xz - 1, xz - 1, y - z \rangle$

$= \langle \overset{f_2}{xz - 1}, \overset{f_3}{y - z} \rangle$.

Is $\{f_2, f_3\}$ a GB for I?

Def 3 Let $f, g \in k[x_1, \ldots, x_n] \setminus \{0\}$, $LM(f) = x^\alpha$ and $LM(g) = x^\beta$.

Let $x^\gamma = LCM(x^\alpha, x^\beta)$ e.g. $LCM(xz, xy) = xyz$.

The <u>S-polynomial</u> or <u>Syzergy polynomial</u> is

$$S(f, g) = \frac{x^\gamma}{LT(f)} \cdot f - \frac{x^\gamma}{LT(g)} \cdot g.$$

E.g. $S(\overset{f}{\underline{xz} - 1}, \overset{g}{\underline{y} - z}) = \left(\frac{xyz}{xz}\right) f - \left(\frac{xyz}{y}\right) \cdot g$

$x^\gamma = xyz$

$$\begin{aligned} &= y(xz - 1) - xz(y - z) \\ &= \cancel{xyz} - y - \cancel{xyz} + xz^2 \\ &= xz^2 - y. \end{aligned}$$

Notice $S(f,g)$ produces a cancellation of the leading terms of f and g s.t. $S(f,g)=0$ or $LM(S(f,g)) < x^\gamma$.

Def 4. Let $f \in k[x_1,\dots,x_n]$. Let $G=\{g_1,\dots,g_t\} \subset k[x_1,\dots,x_n]$.
$\overline{f}^G = f \bmod G$ to be the remainder of $f \div G$.

Theorem 6. (Buchberger's S-polynomial Criterion).

$G=\{g_1,\dots,g_t\}$ is a GB for $I=\langle g_1,\dots,g_t\rangle$ wrt $>$

$\iff$ $S(g_i,g_j) \bmod G = 0$ $\forall i \neq j$.

Example. $S(xz-1, y-z) = xz^2 - y$. $I = \langle \overset{g_1}{xz-1}, \overset{g_2}{y-z} \rangle$.
Using $>_{lex}$ with $x>y>z$.

$$
\begin{array}{ll}
 & a_1 = z \\
 & a_2 = -1 \\
g_1 = \underline{xz}-1 & \overline{)\ \underline{xz^2} - y} \\
g_2 = y-z & -(xz^2 - z) \\
 & \hline \\
 & -y+z \\
 & -(-y+z) \\
 & \hline \\
 & 0 = r
\end{array}
$$

So $G=\{g_1,g_2\}$ is a GB for I.

Theorem. If $I=\langle g_1,\dots,g_s\rangle \subset k[x_1,\dots,x_n]$ and $\gcd(LM(g_i), LM(g_j)) = 1$ for all $i \neq j$ then $G=\{g_1,\dots,g_s\}$ is a GB for I.

Example. Consider the linear system $\{x+y=1, y+z=1, z=1\}$.
Let $I = \langle \overset{g_1}{x+y-1}, \overset{g_2}{y+z-1}, \overset{g_3}{z-1} \rangle$.

In $>_{lex}$ and $>_{grlex}$ with $x>y>z$ $G=\{g_1,g_2,g_3\}$ is a GB for I.

$$[A|b] \quad \begin{array}{ccc} x & y & z \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

is in row Echelon form.

$$\begin{array}{ccc} z & y & x \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

is NOT in row Echelon form.