Buchberger's Algorithm   CLO 2.7.

November 9, 2023   9:23 AM

Let I be an ideal in $k[x_1,\dots,x_n]$ generated by $G=\{f_1,\dots,f_s\}$.

Buchberger's S-polynomial criterion:

$$G \text{ is a GB for } I \iff S(f_i,f_j) \underset{\text{remainder.}}{\underbrace{\bmod G}} = 0 \quad \forall\, i\neq j.$$

Example $I=\langle \overset{f_1}{x^2+y-1},\ \overset{f_2}{xy-2y^2+y}\rangle$ in $>_{lex}$ with $x>y$.

$$S(f_1,f_2) = yf_1 - xf_2 = \cancel{x^2y}+y^2-y-\cancel{x^2y}+2xy^2-xy$$
$$= 2xy^2-xy+y^2-y.$$

$a_1=0$

$a_2=2y-1$

$f_1 = x^2+y-1$

$f_2 = xy-2y^2+y$

$$\begin{array}{r} 2xy^2-xy+y^2-y \\ -(2xy^2-4y^3+2y^2) \\ \hline -xy+4y^3-y^2-y \\ -(-xy+2y^2-y) \\ \hline 4y^3-3y^2 = r \end{array}$$

$$\boxed{S(f_1,f_2) = a_2f_2+r}$$

Since r is not zero, $G=\{f_1,f_2\}$ is NOT a GB for I.

But $S(f_1,f_2) = \boxed{yf_1 - xf_2 = a_1f_1+a_2f_2+r}$

$\Rightarrow\ r = (y-a_1)f_1 + (-x-a_2)f_2 \in I$

$\Rightarrow\ I = \langle f_1,f_2,r\rangle$

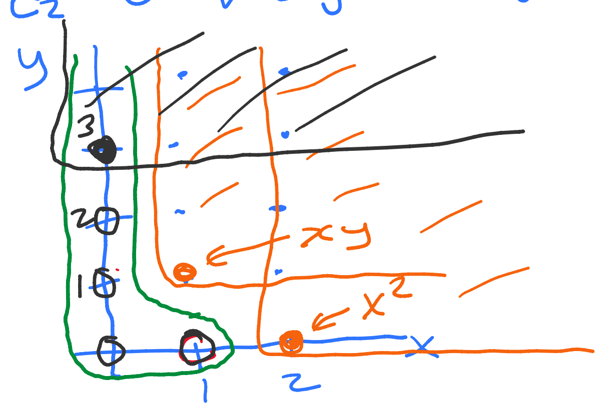
Consider $G_2=\{f_1,f_2,r\}$. Is G_2 a GB for I?

I get $S(f_i,f_j) \bmod G_2 = 0\ \ \forall\, i\neq j$. So yes!

$LM(f_1)=x^2$

$LM(f_2)=xy$

$LM(r)=y^3$



Algorithm Buchberger

Input $F = \{f_1, \dots, f_s\} \subset k[x_1, \dots, x_n] \setminus \{0\}$.

A monomial ordering $<$

Output $G = \{g_1, \dots, g_t\}$ a GB for $I = \langle f_1, \dots, f_s \rangle$.

$G_1 := F$ $\quad k := 1;$

repeat

$k := k+1$. $G_k := G_{k-1}$.

For each pair $\{f, g\} \subset G_{k-1}$ $(f \neq g)$

$r := S(f, g) \bmod G_{k-1}$.

if $r \neq 0$ then $G_k := G_k \cup \{r\}$.

until $G_k = G_{k-1}$.

Since $r \in I$ and $G_k \supset F$, G_k is a basis for I.

If $G_k \neq G_{k-1}$ then $G_k - G_{k-1} = \{r_1, \dots, r_m\}$ for $m > 0$

new remainders.

where $LT(r_i) \notin \langle LT(g) : g \in G_{k-1} \rangle$. Therefore

$$\langle LT(G_1) \rangle \subsetneq \langle LT(G_2) \rangle \subsetneq \langle LT(G_3) \rangle \cdots \subsetneq \langle 1 \rangle$$

This forms an ascending chain of (monomial) ideals in $k[x_1, \dots, x_n]$. By the ACC. it must stabilize, i.e., $G_k = G_{k-1}$ for some $k \geq 1$.

Example 2. Let $I = \langle \overset{f_1}{x^2 + y^2 - 1}, \overset{f_2}{xy - 1} \rangle$ using $>_{lex}$ with $x > y$.

Let $G_1 = \{f_1, f_2\}$.

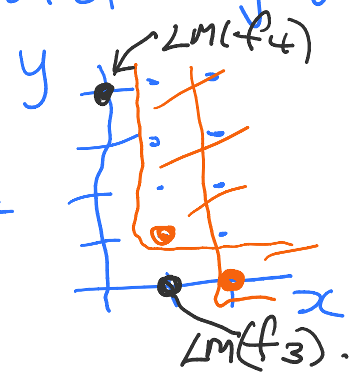
$S(f_1, f_2) = yf_1 - xf_2 = y^3 - y + x = x + y^3 - y \bmod G_1 = x + y^3 - y = f_3$

Let $G_2 = \{f_1, f_2, f_3\}$

I get $S(f_1, f_3) \bmod G_2 = 0$ and

$S(f_2, f_3) \bmod G_2 = -y^4 + y^2 - 1 = f_4$

Let $G_3 = \{f_1, f_2, f_3, f_4\}$.



I get $S(f_i, f_j) \bmod G_3 = 0 \Rightarrow G_3$ is a GB. for I

Thus $\langle LT(I) \rangle = \langle x^2, xy, x, y^4 \rangle = \langle x, y^4 \rangle$

So $G = \{f_3 = x + y^3 - y, f_4 = -y^4 + y^2 - 1\}$ is also a GB for I by def. In general if G is a GB and $LT(g_i) \mid LT(\underline{g_j})$ then $G \setminus \{g_j\}$ is a GB for I.

Def. Let $G = \{g_1, \dots, g_t\}$ be a GB for I wrt $>$.

G is <u>minimal</u> if

(i) $LC(g_i) = 1 \ \forall i$

(ii) $LT(g_i) \nmid LT(g_j) \ \forall i \neq j$. $\Leftarrow$

G is <u>reduced</u> if

(i) $LC(g_i) = 1 \ \forall i$

(ii) $LT(g_i) \nmid$ any term in $g_j \ \forall j \neq i$.

Prop 6. I has a unique reduced GB wrt. $>$.

G is reduced $\Rightarrow$ G is minimal.

Example 3. Let $\langle \overset{f_1}{x+y}, \overset{f_2}{y-1} \rangle$. $G_1 = \{f_1, f_2\}$.

Let $>_{lex}$ with $x > y$.

$S(f_1, f_2) = y f_1 - x f_2 = y^2 + x = x + y^2$

$f_1 = x + y$

$f_2 = y - 1$

$$\begin{array}{r l} a_1 = 1 & \\ a_2 = y & \\ \hline x + y^2 & \\ -(x + y) & \\ y^2 - y & \\ -\ y^2 - y & \\ \hline 0 & \end{array}$$

So $G_1 = \{1 \cdot \underline{x} + y, y - 1\}$ is a minimal GB but not a reduced GB.

Let $f_3 = f_1 - f_2 = (x + y) - (y - 1)$
$= x + 1$.

Thus $I = \langle x + 1, y - 1 \rangle$ and $G = \{x + 1, y - 1\}$ is a reduced GB.