Applications of Groebner bases

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① How can we test if $\{f_1=0, f_2=0, \dots, f_s=0\}$ is inconsistent?

Theorem. Let $I=\langle f_1,\dots,f_s\rangle$ and let $G=\{g_1,\dots,g_t\}$ be a reduced or minimal GB for I wrt $>$. Then

$$G=\{1\} \iff I=k[x_1,\dots,x_n]$$

Proof. ($\Rightarrow$) $G=\{1\} \Rightarrow 1\in I \Rightarrow \forall h\in k[x_1,\dots,x_n].\ h\cdot 1\in I = k[x_1,\dots,x_n]$.

($\Leftarrow$) $I=k[x_1,\dots,x_n] \Rightarrow 1\in I \Rightarrow \underline{1}\in \underline{\langle LT(I)\rangle}$

G is a GB for $I \Rightarrow \exists g\in G$ s.t. $\underline{LT(g)} \mid 1 \Rightarrow g\in k\setminus\{0\}$.

G is reduced or minimal $\Rightarrow g=1. \Rightarrow 1\in G$.

If $|G|>1$ then $\exists h\in G,\ h\neq 1$.

But $LT(g)=\underline{1} \mid \underline{LT(h)} \Rightarrow G$ is not minimal and not reduced $\boxtimes$

Thus $G=\{1\}$.

If $G=\{1\} \Rightarrow I=\langle 1\rangle$

$\Rightarrow \mathbb{V}(f_1,\dots,f_s) \overset{\text{prop 4}}{=} \mathbb{V}(1)=\emptyset$.

Question. if $\mathbb{V}(f_1,\dots,f_s)=\emptyset$ is $G=\{1\}$? Not in general.

$\mathbb{V}(\underline{x^2+1})=\emptyset$ in $\mathbb{R}[x]$ but

$G=\{1\cdot x^2+1\}$ is the reduced GB for $I=\langle x^2+1\rangle$.

CLO 4.1 <u>Hilbert's Weak Nullstellensatz.</u>

If k is algebraically closed then $\mathbb{V}(f_1,\dots,f_s)=\emptyset$

e.g. $k=\mathbb{C}$ $\Rightarrow 1\in I \Rightarrow G=\{1\}$.

inconsistent.

② <u>Solving systems of polynomial equations.</u> CLO 3.1.

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For $0\le i<n$ define the ith elimination ideal

Def 1. Let I be an ideal in $k[x_1,\dots,x_n]$.
For $0 \le j \le n$ define the j'th elimination ideal
$$I_j = I \cap k[x_{j+1},\dots,x_n].$$
$I_j = \{ f \in I$ where $x_1,\dots,x_j$ have been elimination$\}$.
Exercise. Show that I_j is an ideal in $k[x_{j+1},\dots,x_n]$.

Theorem 2 (the elimination theorem). Let $G = \{g_1,\dots,g_t\}$ be a GB for I under $>_{lex}$ with $x_1 > x_2 > \dots > x_n$. Then
$$G_j = G \cap k[x_{j+1},\dots,x_n] \text{ is a GB for } I_j$$

Example. $I = \langle \underbrace{x^2+y^2-z}_{f_1}, \underbrace{xy-1}_{f_2} \rangle \subset \mathbb{Q}[x,y]$
$G = \{ \underbrace{\underline{x}+y^3-2y}_{g_1}, \underbrace{y^4-2y^2+1}_{g_2} \}$ is a GB for I in $>_{lex}$ with $x>y$.

$G_1 = G \cap \mathbb{Q}[y] = \{g_2\}$ is a GB for $I \cap \mathbb{Q}[y] = \langle y^4 - 2y^2+1 \rangle$.

The y-coordinates of $V(f_1,f_2) = V(g_1,g_2)$ are the 4 solutions $0 = y^4-2y^2+1 = (y^2-1)^2 \Rightarrow y = \pm 1$.
Now $g_1 = x + y^3 - 2y = 0 \Rightarrow$ $y=1$, $x=1$
$y=-1$, $x=-1$.

Proof. We must show $\langle LT(G_j) \rangle = \langle LT(I_j) \rangle$
($\subset$) By construction $G_j \subset I_j \Rightarrow \langle LT(G_j) \rangle \subset \langle LT(I_j) \rangle$.
($\supset$). Let $f \in I_j$, $f \neq 0$. $f \in I_j \Rightarrow f \in I$.
G is a GB $\Rightarrow \exists g \in G_j$ s.t. $LT(g) \mid LT(f)$.
$f \in I_j \Rightarrow LT(f) \in k[x_{j+1},\dots,x_n]$.
$\Rightarrow LT(g) \in k[x_{j+1},\dots,x_n]$.
In $>_{lex}$ with $x_1 > x_2 > \dots > x_n$ $LT(g) \in k[x_{j+1},\dots,x_n]$
$\Rightarrow g \in k[x_{j+1},\dots,x_n] \Rightarrow g \in G_j$.

$\Longrightarrow \; g \in k[x_{j+1}, \dots, x_n] \; \Rightarrow \; g \in G_j$.

Hence $\langle LT(I_j) \rangle \subset \langle LT(G_j) \rangle$.

③ What does a GB tell us about $|\mathbb{V}(f_1, \dots, f_s)|$?

Example. Let $I = \langle x^2+y-1, xy-2y^2+y \rangle$. We found

>lex with $x>y$.

$G = \{ \overset{f_1}{x^2+y-1}, \overset{f_2}{xy-2y^2+y}, \overset{f_3}{4y^3-3y^2} \}$ is a GB for I.

$G = \{ \overset{f_1}{x^2+y-1}, \overset{f_2}{xy-2y^2+y}, \overset{f_3}{y^3-\tfrac{3}{4}y^2} \}$ " " " " minimal ✓ reduced ✓

The generators x^2+y-1 and $y^3-\frac{3}{4}y^2$ tell us $|\mathbb{V}(f_1,f_2,f_3)| \leq 6$.

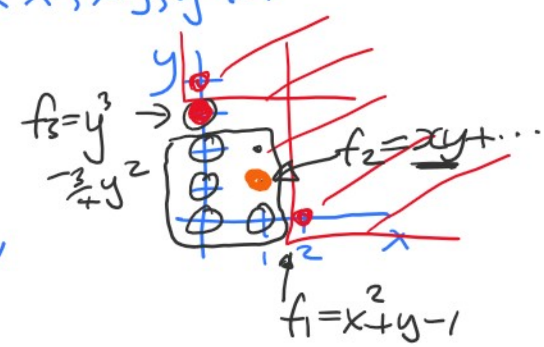
How does f_2 constrain the $\mathbb{V}(f_1,f_3)$?

CLO 5.3. Let $\overline{\langle LT(I) \rangle} = \{ x^\alpha : x^\alpha \notin \langle LT(I) \rangle \}$.

In our example $\langle LT(I) \rangle = \langle LT(f_1), LT(f_2), LT(f_3) \rangle$

$= \langle x^2, xy, y^3 \rangle$.

$\overline{\langle LT(I) \rangle} = \{1, x, y, y^2\}$.



If $f \in \mathbb{Q}[x,y]$ then f mod G will be a $\mathbb{Q}$-linear combination of $\{1, x, y, y^2\}$.

Prop 8 says $|\mathbb{V}(f_1,f_2)| \leq |\overline{\langle LT(I) \rangle}| = 4$.