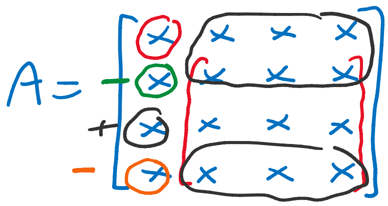
Determinants

September 26, 2023 9:05 AM

Computational Exact Linear Algebra.

Let A be an $n \times n$ matrix over a comm. ring R.
How fast can we compute $\det(A)$?

Def (cofactor expansion)



Let $B_{ij} = A$ with row i and col j removed.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+1} A_{i1} \cdot \det(B_{i1})$$

$$\det([a]) = a.$$

Observe $\det(A) \in R$.

Let $M(n) = $ # mults in R that this method does.

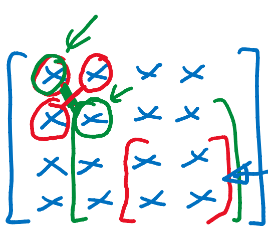
$M(1) = 0.$

$M(n) = n \cdot M(n-1) + n$

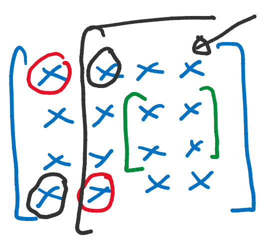
$M(2) = 2 \cdot 0 + 2.$

$M(n) = 1.72 \cdot n!$ mults in R.

Gentleman & Johnson (1976)



det is computed twice.



$\binom{4}{2}$ 2×2 submatrices from the last two columns.

① Compute all $\binom{n}{2}$ 2×2 determinants from columns $n, n-1$ using the n 1×1 determinants from column n.

② Compute all $\binom{n}{3}$ 3×3 determinants from the last 3 columns (picking 3 rows from n) using the dets from step 2.

Repeat this until we compute all $\binom{n}{n} = 1$ dets of all A.

Let $E(n) = $ # mults in R that the G&J method does.

$$G(n) = \sum_{i=2}^{n} \binom{n}{i} \cdot i = \tfrac{1}{2} n \cdot 2^n - n \in O(n \cdot 2^n)$$ which is exponential in n.

each

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i×i minors from last i cols

i mults each

$$\frac{M(10)}{G(10)} = \frac{6{,}255{,}300}{5{,}110} = 1220.$$

Can we compute $\det(A)$ in $O(n^3)$ ring ops, $+, -, \times$?

Open.

Berkowitz (1984) $O(n^4)$

Kaltofen (1992) $O(n^{3.5})$.

What if R is an integral domain? E.g. $\mathbb{Z}$, $\mathbb{Z}[x_1, \ldots, x_n]$.

We could run G.E. in the fraction field. $\mathbb{Q}$ $\mathbb{Z}(x_1, \ldots, x_n)$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ 5 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 - \frac{5}{2}R_1]{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5/2 & -7/2 \\ 0 & 1/2 & -13/2 \end{bmatrix} \xrightarrow[\frac{1/2}{-5/2}R_2 = +\frac{1}{5}R_2]{R_3 \leftarrow R_3 -} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -5/2 & -7/2 \\ 0 & 0 & -\frac{36}{5} \end{bmatrix}$$

So $\det(A) = \cancel{2} \cdot \frac{-5}{\cancel{2}} \cdot \frac{-36}{\cancel{5}} = +36.$

Can we avoid fractions?

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 1 \\ 5 & 3 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow 2R_3]{R_2 \leftarrow 2R_2} \overset{B}{\begin{bmatrix} 2 & 1 & 3 \\ 6 & -2 & 2 \\ 10 & 6 & 2 \end{bmatrix}} \xrightarrow[R_3 \leftarrow R_3 - 5R_1]{R_2 \leftarrow R_2 - 3R_1} \overset{C}{\begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 1 & -13 \end{bmatrix}}$$

$$\xrightarrow{R_3 \leftarrow -5R_3} \overset{D}{\begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & -5 & +65 \end{bmatrix}} \xrightarrow{R_3 \leftarrow R_3 - 1 \cdot R_2} \overset{E}{\begin{bmatrix} 2 & 1 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & 72 \end{bmatrix}}$$

$$\det(E) = \det(D) = -5\det(C) = -5\det(B) = -5 \cdot 4 \cdot \det(A).$$

$$\det(E) = 2 \cdot (-5) \cdot 72.$$

$$\det(A) = \det(E)/(-20) = \frac{2(-5) \cdot 72}{-5 \cdot 4} = \frac{72}{2} = 36.$$

The only divisions are exact because $\det(A) \in \mathbb{Z}$.

$-5 \cdot 4$

The only divisions are exact because $\det(A) \in \mathbb{Z}$.

<u>But</u> the size of the integers grows exponentially in n.

How big is $\det(A)$? Where $A_{ij} \in \mathbb{Z}$.

Hadamards bound: $|\det(A)| \leq \prod_{i=1}^{n} \sqrt{\sum_{j=1}^{n} A_{ij}^2}$

If $|A_{ij}| < \underset{\overset{\shortparallel}{10}}{B}^m$ then $|\det(A)| < \prod_{i=1}^{n} \sqrt{n \cdot B^{2m}} = n^{n/2} \cdot B^{nm}$

The size of $\det(A) \leq \lceil \log_B B^{nm} n^{n/2} \rceil =$

$= mn + \lceil \frac{n}{2} \underbrace{\log_B n}_{B=2^{64}} \rceil \leq (m+1)n$ digits base B.