Bareiss/Edmonds Fraction Free Gaussian Elimination

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Example

$$-5 = \det\left(\begin{bmatrix}2&1\\3&-1\end{bmatrix}\right)$$

$$\begin{bmatrix}2&1&3\\3&-1&1\\5&3&1\end{bmatrix} \xrightarrow[R_3 \leftarrow (2R_3-5R_1)/1]{R_2\leftarrow(2R_2-3R_1)/1} \begin{bmatrix}2&1&3\\0&-5&-7\\0&1&-13\end{bmatrix} \xrightarrow[2]{R_3\leftarrow \boxed{-5R_3-1\cdot R_2}} \begin{bmatrix}2&1&3\\0&-5&-7\ \checkmark\\0&0&\frac{72}{2}=36\end{bmatrix}$$

$36 = \det(A)$

We did $O(n^3)$ ops in R $(+, -, \times,$ and exact $\div)$

Proof that the division by $A^{(k-1)}_{k-2,k-2}$ is exact in R for $n=3$.

$$A = \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} \xrightarrow[R_3\leftarrow aR_3-gR_1]{R_2\leftarrow aR_2-dR_1} \begin{bmatrix}a&b&c\\0&\overset{S}{ae-db}&\overset{T}{af-dc}\\0&\underset{U}{ah-gb}&\underset{V}{ai-gc}\end{bmatrix} = B$$

$$= \begin{bmatrix}a&b&c\\0&S&T\\0&U&V\end{bmatrix} \xrightarrow[\;=\frac{S}{a}R_3-\frac{U}{a}R_2\;]{R_3\leftarrow \frac{SR_3-UR_2}{a}} \begin{bmatrix}a&b&c\\0&S&T\\0&0&\frac{SV-TU}{a}\end{bmatrix} = C.$$

Does $a \mid SV-TU$.

$$\det(C) = \frac{S}{a}\det(B) = a^2\frac{S}{a}\det(A) = \underline{aS\det(A)}.$$

$$\det(C) = a\cdot S\cdot\frac{SV-TU}{a} = S(SV-UT).$$

$$\underset{\rightarrow R.}{\det(A)} = \frac{\det(C)}{aS} = \frac{S(SV-UT)}{aS} = \frac{\overset{R\ R\ \ R\ R}{SV-UT}\in R}{a\in R}.$$

Since $\det(A)\in R$ then $a \mid SV-UT$.

Check $SV-UT = (ae-db)(ai-gc)-(ah-gb)(af-dc)$

$$= \underline{a}^2ei - \underline{a}idb - \underline{a}egc + \cancel{dbgc} - \underline{a}^2hf + \underline{a}fgb + \underline{a}hdc - \cancel{gbdc}.$$

The "Achilles Heal" of the Bareiss/Edmonds algorithm.

$$\begin{array}{c|c} & \begin{array}{cccc} & & k & j\end{array}\\ \begin{array}{c} \\ \\ k\rightarrow \\ i\rightarrow\end{array} & \begin{bmatrix}\times&\times&\times&\times\\0&\times&\times&\times\\0&0&\times&\times\\0&0&\times&\bullet\end{bmatrix}\end{array}$$

$$A^{(k)}_{ij} = \frac{A^{(k-1)}_{kk}\cdot A^{(k-1)}_{ij} - A^{(k-1)}_{ik}A^{(k-1)}_{kj}}{A^{(k-2)}_{k-2,k-2}}$$

At step $k=n-1$

$$A^{(n-1)}_{nn} = \frac{A^{(n-2)}_{n-1,n-1}\cdot A^{(n-2)}_{nn} - A^{(n-2)}_{n,n-1}A^{(n-2)}_{n-1,n}}{\cdots} \quad = N$$

At step $k=n-1$, $i=n$, $j=n$

$$\underset{\substack{\uparrow\\ \det(A)}}{A_{nn}^{(n-1)}} = \frac{A_{n-1n-1}^{(n-2)} \cdot A_{nn}^{(n-2)} - A_{nn-1}^{(n-2)} A_{n-1n}^{(n-2)}}{A_{n-2n-2}^{(n-3)}}$$

$A_{n-2n-2}^{(n-3)} \leftarrow$ det of the $n-2 \times n-2$ principle minor of A.

$$N = \underset{\substack{\uparrow\\ \det(A)}}{A_{nn}^{(n-1)}} \cdot \underset{\substack{\uparrow\\ \det(\ )}}{A_{n-2n-2}^{(n-3)}}.$$

For $R = \mathbb{Z}[x_1, \ldots, x_n]$ the numerator N is a product of two polynomials $\det(A)$ and $\det(\ )$. It can be much bigger than $\det(A)$.

$$S_3 = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

How fast can we test if $\det(A) = 0$? $\qquad \alpha = (1, 0, 1, 2)$.

Try $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$.

$$T_4 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 & x_2 \\ x_4 & x_3 & x_2 & x_1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 1 & -2 & -3 \\ & 0 & -2 & -4 \end{bmatrix}$$

So $A \sim I \Rightarrow \det(A) \neq 0 \Rightarrow \det(T) \neq 0$.

The Schwarz-Zippel Lemma (1978).

Let $D = \mathbb{Z}$ be an integral domain, $f \in D[x_1, \ldots, x_n]$, $f \neq 0$. (f ← $\det(A)$.)

Let S be a finite subset of D.

Suppose α is chosen at random from S. Then

$$\text{Prob}[f(\alpha) = 0] \leq \frac{\deg(f)}{|S|} \leftarrow \text{total degree}$$

Let $f = \det(T_4) \in \mathbb{Z}[x_1, x_2, x_3, x_4] = D$.

$\deg(f) \leq 4$. Take $S = [0, 10^6)$. $|S| = 10^6$

$$\text{S-W} \Rightarrow \text{Prob}[f(\alpha) = 0] \leq \frac{4}{10^6} = \frac{1}{250{,}000}.$$

Pick $\beta \in S$ at random.

$$\text{Prob}[f(\alpha)=0 \wedge f(\beta=0)] \leq \frac{4}{10^6} \cdot \frac{4}{10^6} = \frac{16}{10^{12}}.$$

Schwarz. Do k random α's until

$$\text{Prob}[f(\alpha_k)=0] \leq \frac{1}{10^{100}}.$$

Proof (by induction on n) Wikipedia.

Case $n=1$. $f \in D[x_1]$ ← int dom. $\Rightarrow$ # roots $\leq \deg(f)$.

$F[x_1]$

$$\text{Prob}[f(\alpha)=0] \leq \frac{\deg(f)}{|S|}. \quad \checkmark$$