Adjugates and Characteristic Polynomials

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Assignment #2 due next Tuesday @ 11pm.
Monday office hour moved to Tuesday.

Let R be a comm. ring and $A \in R^{n\times n}$.
How fast can we compute $\det(A)$ and $\det(A-\lambda I)$?

<u>The adjugate matrix (adjoint) adj(A).</u>

Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $A^{-1} = \frac{1}{ad-bc}\cdot \underbrace{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{\mathrm{adj}(A)} = \det(A)^{-1}\cdot \mathrm{adj}(A)$.

Def. Let A_{ij} denote the $n-1 \times n-1$ submatrix of A obtained by deleting row i and column j. Let $C = \mathrm{cof}(A)$ where $C_{ij} = (-1)^{i+j}\cdot \det(A_{ij})$. Define $\mathrm{adj}(A) = C^T$.

E.g. $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$C_{11} = (-1)^{1+1}\cdot \det([d]) = d$.
$C_{12} = (-1)^{1+2}\cdot \det([c]) = -c$
$C_{21} = (-1)^{2+1}\cdot \det([b]) = -b$
$C_{22} = (-1)^{2+2}\cdot \det([a]) = a$.

$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$ $\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Properties of adj(A).

① $\mathrm{adj}(AB) = \mathrm{adj}(B)\cdot \mathrm{adj}(A)$
② if $\det(A) \neq 0$ then $\mathrm{adj}(A) = \det(A)\cdot A^{-1}$

Proof ① using ②

R is commutative and $\det(A), \det(B) \in R$

$$\begin{aligned} \mathrm{adj}(AB) &= \det(AB)\cdot (AB)^{-1} \\ &= \det(A)(\det(B)\cdot B^{-1}) A^{-1} \\ &= (\det(B)\cdot B^{-1})(\det(A)\cdot A^{-1}) \\ &= \mathrm{adj}(B)\cdot \mathrm{adj}(A). \end{aligned}$$

Proof of ②. One way to compute A^{-1} is to solve $AX = I$ for X, i.e.,

$A\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} \Rightarrow A x_j = e_j$. Apply Cramer's rule

Let $A^{(i)} = \begin{bmatrix} | & & | & & | \\ A_1 & \cdots & e_j & \cdots & A_n \\ | & & | & & | \end{bmatrix}$. $x_{ij} = \frac{\det(A^{(i)})}{\det(A)}$

col i row i x_j

$$\det(A(i)) = \det\left(\begin{array}{c} \\ \left[\begin{matrix} | & & 0 & & | \\ A_1 & \cdots & 1 & \cdots & A_n \\ | & & 0 & & | \\ & & \vdots & & \\ & & 0 & & \end{matrix}\right] \\ \text{col } i \end{array}\right) \leftarrow \text{row } j = (-1)^{i+j}\det(\bar{A}_{ji}) = C_{ji}.$$

$$x_{ij} = \frac{C_{ji}}{\det(A)} \Rightarrow X = A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}.$$

The characteristic polynomial $c(\lambda)$.

Let $A \in R^{n\times n}$. Define $C(\lambda) = \det(A - \lambda I) = \sum_{i=0}^{n} c_i \lambda^i \in R[\lambda]$.

The eigenvalues of A are the roots of $C(\lambda)$.

The Berkowitz algorithm computes $C(\lambda)$ using $O(n^4)$ ring operations $+,-,\times$.

NB: $c_0 = C(\lambda=0) = \det(A - 0\cdot I) = \det(A)$.

E.g. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A-\lambda I) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right)$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \underbrace{1}_{c_2}\cdot\lambda^2 + \underbrace{(-a-d)}_{c_1}\lambda + ad - bc \qquad c_0 = \det(A).$$

Let $A = \begin{bmatrix} \begin{bmatrix} A_r \end{bmatrix} & S \\ -R- & A_{nn} \end{bmatrix}$ where $A_r \in R^{r\times r}$ and $S, R^T \in R^r$

$r = n-1$

Theorem 1. $\det(A) = \det(A_r)A_{nn} - R^T\cdot(\operatorname{adj}(A_r)\cdot S) \in R$.

Check.

$$\det\left(\begin{bmatrix} a & b & \boxed{e} \\ c & d & \boxed{f} \\ \boxed{g \ h} & & i \end{bmatrix}\right) = a\begin{vmatrix} d & f \\ h & i \end{vmatrix} - c\begin{vmatrix} b & e \\ h & i \end{vmatrix} + g\begin{vmatrix} b & e \\ d & f \end{vmatrix}$$

$$= adi - ahf - cbi + che + gbf - ged.$$

RHS: $(ad-bc)i - \left(\overset{R^T}{\begin{bmatrix} g \\ h \end{bmatrix}}\cdot\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\cdot\overset{S}{\begin{bmatrix} e \\ f \end{bmatrix}}\right) = \begin{bmatrix} g \\ h \end{bmatrix}\cdot\begin{bmatrix} de-bf \\ -ce+af \end{bmatrix}\right)$

$$= adi - bci - gde + gbf + hce - haf$$

Theorem 2. Let $c_r(\lambda) = \det(A_r - \lambda I) = \sum_{i=0}^{r} c_i\lambda^i$.

$$\operatorname{adj}(A_r - \lambda I) = -\sum_{k=1}^{r}\sum_{j=0}^{r-k} c_{k+j}A_r^j\lambda^{k-1}. \qquad r = n-1.$$

$\mathrm{adj}(A_r - \lambda I) = - \sum_{k=1} \sum_{j=0} c_{k+j} \cdots$

$A_r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ r

$\mathrm{adj}\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right) = \begin{bmatrix} d-\lambda & -b \\ -c & a-\lambda \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \lambda.$

RHS. $\underset{\substack{k=1 \\ j=0}}{-C_1 \cdot I \cdot \lambda^0} \quad \underset{\substack{k=1 \\ j=1}}{- C_2 A_r \cdot 1} \quad \underset{\substack{k=2 \\ j=0}}{- C_2 I \cdot \lambda^1}$

$r=2$

$= - \begin{bmatrix} -a-d & 0 \\ 0 & -a-d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \lambda$

$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \lambda.$:)

Let $A \in R^{n \times n}$. Apply Th. 1. to $A - \lambda I$.

$C_n(\lambda) = \det(A - \lambda I) = \det(A_r - \lambda I_r)(A_{nn} - \lambda) - R^T \cdot (\mathrm{adj}(A_r - \lambda I_r) \cdot S)$

recursively

$A - \lambda I = \begin{bmatrix} \boxed{A_r - \lambda I_r} & S \\ - R - & A_{nn} - \lambda \end{bmatrix} = C_r(\lambda)(A_{nn} - \lambda) - R^T \cdot \left(- \sum_{k=1}^{r} \sum_{j=0}^{r-k} C_{k+j} A_r^j \lambda^{k-1}\right) S$

$= C_r(\lambda)(A_{nn} - \lambda) + \sum_{k=1}^{r} \sum_{j=0}^{r-k} C_{k+j} \cdot \underbrace{(R^T \cdot A_r^j \cdot S)}_{= Q_j \in R.} \lambda^{k-1}$ compute once.

$r = n-1$ recursive call.

$0 \le j \le r-1$

$Q_j = R^T (A_r^j \cdot S).$

$B \leftarrow I_r$
$Q_0 \leftarrow R^T \cdot S$ $\quad r$ mults.
for $j = 1, 2, \ldots, r-1$ do
$\quad B \leftarrow A_r \cdot B$ $\quad r^3$ mults
$\quad Q_j \leftarrow R^T \cdot B \cdot S$ $\quad r^2 + r$

Total $\quad r + (r-1)(r^3 + r^2 + r) \in O(r^4)$

Good method. $R^T (A_r^{j-1} (A_r \cdot S))$

$Q_0 \leftarrow R^T \cdot S.$ $\quad r$ mults
for $j = 1, 2, \ldots, r-1$ do
$\quad S \leftarrow A_r \cdot S$ // $= A_r^j \cdot S$ $\quad r^2$ mults
$\quad Q_j \leftarrow R^T \cdot S.$ $\quad r$ mults

Total: $(r-1)(r^2 + r) + r \in O(r^3)$.
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