The Berkowitz Algorithm

October 2, 2023 9:16 PM

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Algorithm Berkowitz.
Input  $A \in R^{n\times n}$ where $R$ is a comm. ring.
Output $C(\lambda) = \det(A - \lambda I) \in R[\lambda]$.
  if $n=1$ output $A_{11} - \lambda$   // $[A_{11} - \lambda]$
  Let $A = \begin{bmatrix} A_r & | \ S \\ - R - & A_{nn} \end{bmatrix}$
  $r = n-1$.

  Compute $C_r(\lambda) = \text{Berkowitz}(A_r) = \sum_{i=0}^{r} c_i \lambda^i$.
    $Q_0 \leftarrow R^T \cdot S$                 $r$
    for $i = 1, 2, \dots, r-1$ do
        $S \leftarrow A_r \cdot S$               $r^2$
        $Q_j \leftarrow R^T \cdot S$             $r$
    $C_n(\lambda) \leftarrow C_r(\lambda)(A_{nn} - \lambda) + \sum_{k=1}^{r} \sum_{j=0}^{r-k} C_{k+i} \cdot Q_j \lambda^{k-1}$
    Output $C_n(\lambda)$.
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E.g.

$$A = \left[\begin{array}{c|cc|c} & \multicolumn{2}{c}{A} & S \\ & 1 & 2 & 3 \\ & 1 & 0 & 4 \\ R & 3 & 1 & 2 \end{array}\right]$$

$$A_r = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad C_r(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 1 & 0-\lambda \end{bmatrix}\right)$$

$$= 1\lambda^2 - \lambda - 2.$$

C_2 $\quad C_1 = -1 \quad C_0 = -2.$

$$Q_0 = R^T \cdot S = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 13$$

$$Q_1 = R^T A_r S = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 11 \\ 3 \end{bmatrix} = 36.$$

$$C(\lambda) = (\lambda^2 - \lambda - 2)(2 - \lambda) + \underset{\substack{k=1 \\ j=0}}{\overset{=-1}{C_1}} \cdot 13 \cdot + \underset{\substack{k=1 \\ j=1}}{\overset{=1}{C_2}} \cdot 36 \cdot 1 + \underset{\substack{k=2 \\ j=0}}{\overset{=1}{C_2}} \cdot 13 \cdot \lambda$$

$$= 2\lambda^2 - \lambda^3 \underline{- 2\lambda} + \lambda^2 - 4 + \underline{2\lambda} - 13 + 36 + 13\lambda,$$

$$= -\lambda^3 + 3\lambda^2 + 13\lambda + 19.$$

Cost: Let $M(n)$ be the #mults in R that algorithm Berkowitz does.

$$M(n) = \underset{\overset{\uparrow}{\text{recursive call}}}{M(n-1)} + r + \underline{(r-1)(r^2+r)} + \underbrace{\sum_{k=1}^{r}\sum_{j=0}^{r-k} 1}_{} = \tfrac{1}{2}r^2 + \tfrac{1}{2}r.$$

$$M(n) = M(n-1) + r^3 + \tfrac{1}{2}r^2 + \tfrac{1}{2}r$$

$$M(1) = 0$$

$$M(n) = \tfrac{1}{4}n^4 - \tfrac{1}{3}n^3 + \dots \in O(n^4) \text{ mults in } R.$$

Remark: If A is sparse (# non-zero entries in $A \ll \underline{n^2}$).

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$Q_0 \leftarrow R^T S$            $r$
for $i = 1, 2, \dots, r-1$ do
    $S \leftarrow A_r \cdot S$          $3r$
    $Q_j \leftarrow R^T \cdot S$        $r$
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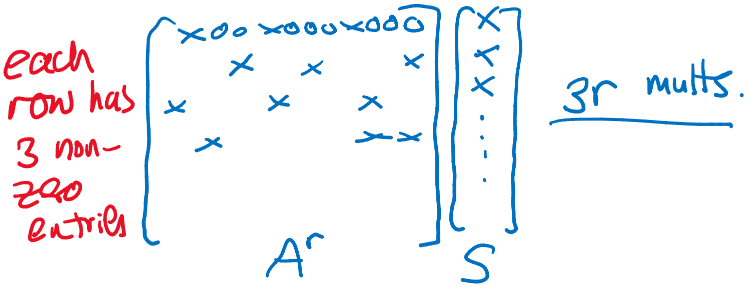
$r + (r-1)(4r) \in O(r^2)$.

If A is sparse $\Rightarrow$ $O(n^3)$ mults in R.



To compute det(A) over R.

Case R a field: Ordinary Gaussian elimination does $\frac{1}{3}n^3 + \dots$ mults in R.

Case R an int dom: Bareiss/Edmonds does $\frac{\boxed{2}}{3}n^3 + \dots$ mults in R. and $\frac{1}{3}n^3 + \dots$ exact ÷ in R.

$$A_{ij} \leftarrow \frac{A_{kk}\cdot A_{ij} - A_{kj}\cdot A_{ik}}{A_{k-1\,k-1}} \uparrow$$

Case R a comm ring: Berkowitz $\frac{1}{4}n^4 + \dots$ mults in R.

Kaltofen's. $O(n^{3.5})$ mults in R.