Q1

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(a). (i) Since I and J are ideals, $0 \in I$ and $0 \in J \Rightarrow 0 \in I \cap J$.

(ii) Let $a, b \in I \cap J \Rightarrow a, b \in I \Rightarrow a+b \in I$
$\Rightarrow a, b \in J \Rightarrow a+b \in J$ $\Big\} \Rightarrow a+b \in I \cap J$.

(iii) Let $a \in I \cap J$, $h \in k[x_1, \dots, x_n]$ $\Rightarrow$ $a \in I \Rightarrow ah \in I$, $a \in J \Rightarrow ah \in J$ $\Big\} \Rightarrow ah \in I \cap J$.

So $I \cap J$ is an ideal.

(b) $I = \langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_t \rangle = J$.

$I = J \Rightarrow g_i \in I \Rightarrow g_i = \sum_{j=1}^{s} h_{ji} f_j$ for some $h_{ji} \in k[x_1, \dots, x_n]$

Let $a \in V(f_1, \dots, f_s)$. $\Rightarrow f_i(a) = 0$

$\Rightarrow \sum_{j=1}^{s} h_{ji}(a) \cdot f_j(a) = 0$ –

$\Rightarrow g_i(a) = 0 \Rightarrow a \in V(g_1, \dots, g_t)$.

(c). $I = \langle \overset{f_1}{x-z}, \overset{f_2}{xy-1}, \overset{f_3}{y-z} \rangle$

$f_2 - y f_1 = xy - 1 - (xy - yz) = yz - 1 = f_3$

$\overset{\text{VUL}}{\Rightarrow}$ $I = \langle \overset{f_1}{x-z}, \overset{f_3}{yz-1}, \overset{f_2}{y-z} \rangle$

VUL $f_3 - z f_2 = yz - 1 - yz + z^2 = z^2 - 1 = f_4$

$\Rightarrow$ $I = \langle x - z, z^2 - 1, y - z \rangle$.

This is a GB under $>_{lex}$ with $x > y > z$. note.

Now $V(f_1, f_2, f_3) = V(f_1, f_5, f_3) = \{(1,1,1), (-1,-1,-1)\}$.