Q2

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(a) I think the cleanest proofs are by contradiction.

Prove $<$ is a well ordering $\iff$ 1 is the least monomial in $<$.

($\Rightarrow$) Suppose $<$ is a well ordering.

TAC suppose $1 > x^\alpha$ for some $\alpha \in \mathbb{Z}^n_{\geq 0}$.

Then $x^\alpha > x^\alpha \cdot x^\alpha = x^{2\alpha}$ by (ii).

And $x^\alpha \cdot x^\alpha > x^\alpha \cdot x^{2\alpha}$ by (ii)

$\Rightarrow x^{2\alpha} > x^{3\alpha}$.

We have $1 > x^\alpha > x^{2\alpha} > x^{3\alpha}$

Continuing this we get $1 > x^\alpha > x^{2\alpha} > x^{3\alpha} > x^{4\alpha} > \cdots$

contradicting every subset of monomials has a least element.

($\Rightarrow$) Another proof. Suppose $<$ is a well ordering.

Then $S = \mathbb{Z}_{\geq 0}$ has a least element say α.

So $x^\alpha \leq x^\beta$ for all $\beta \in \mathbb{Z}_{\geq 0}$.

Suppose $x^\alpha \neq 1 \Rightarrow x^\alpha < 1 \Rightarrow x^\alpha(x^\alpha) < x^\alpha \cdot 1$ by (ii)

$\Rightarrow x^{2\alpha} < x^\alpha$ $\boxtimes$

So 1 is the smallest element of $\mathbb{Z}_{\geq 0}$.

($\Leftarrow$) Suppose 1 is the least monomial.

Let S be a subset of $\mathbb{Z}^n_{\geq 0}$, $S \neq \{\}$.

Let $I = \langle x^\alpha : \alpha \in S \rangle$.

Dickson's Lemma $\Rightarrow I = \langle x^{\alpha(1)}, x^{\alpha(2)}, \ldots, x^{\alpha(s)} \rangle$ for some $s \in \mathbb{N}$.

[Doesn't Dickson's Lemma's proof assume a well ordering?]

WLOG suppose $x^{\alpha(1)} < x^{\alpha(2)} < \cdots < x^{\alpha(s)}$.

Claim $x^{\alpha(1)}$ is the least element of S.

TAC suppose $x^\beta \in S$ with $x^\beta < x^{\alpha(1)}$.

$\Rightarrow x^\beta \in I \Rightarrow x^{\alpha(i)} \mid x^\beta$ for some $1 \leq i \leq s$.

If $i = 1$ we have $x^{\alpha(1)} \mid x^\beta \Rightarrow x^{\alpha(1)} \leq x^\beta$ $\boxtimes$

If $i > 1$ we have $x^{\alpha(1)} < x^{\alpha(i)} \leq x^\beta$ $\boxtimes$

Let S be a subset of $\mathbb{Z}_{\geq 0}$, $S \neq \emptyset$.
If $0 \in S$ then since $0 \leq \alpha$, S has a least element 0.
If $0 \notin S$ then consider $S' = S \cup \{0\}$.
S' has a least element 0 and $>$ is a total ordering. so
$\exists\, \beta \in S$ s.t. $0 < \beta < \alpha$ in S' for all $\alpha \in S \setminus \{0\}$.
Hence $\beta \in S'$ is the least element in S.
Thus $>$ is a well ordering.

(b) $>_{lex}$ with $x > y$

$f_1 = x + y^2 - 1$
$f_2 = xy - 1$

$$
\begin{array}{r l}
 & a_1 = 2y \\
 & a_2 = 0 \\
\hline
 & \big) \; 2xy + y^3 - y - 1 = f \\
 & -(2xy + 2y^3 - 2y) \\
\hline
 & -y^3 + y - 1 = r
\end{array}
$$

So $f = 2y f_1 + 0 f_2 + (-y^3 + y - 1)$.

$>_{grlex}$ with $x > y$

$f_1 = y^2 + x - 1$
$f_2 = xy - 1$

$$
\begin{array}{r l}
 & a_1 = y \\
 & a_2 = 1 \\
\hline
 & \big) \; y^3 + 2xy - y - 1 = f \\
 & -(y^3 + xy - y) \\
\hline
 & = xy - 1 \\
 & - xy - 1 \\
\hline
 & = 0 = r
\end{array}
$$

So $f = y f_1 + f_2 \Rightarrow f \in \langle f_1, f_2 \rangle$.