FFT 2 algorithm

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Ref MCA 8.2 p. 230

Let $f(x) = (a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{n/2-1}) + (a_{\frac{n}{2}} x^{n/2} + \cdots + a_{n-1} x^{n-1})$

(1) $f \div x^{n/2} - 1$ $\quad \boxed{f(x) = q_0(x)(x^{\frac{n}{2}} - 1) - r_0(x)}$ $\quad r_0(x) = f(x^{n/2} = 1)$

$r_0(x) = (a_0 + a_{n/2}) + (a_1 + a_{\frac{n}{2}+1})x + \cdots + (a_{\frac{n}{2}-1} + a_{n-1}) x^{n/2-1}$

(2) $f \div x^{n/2} + 1$ $\quad f(x) = q_1(x)(x^{\frac{n}{2}} + 1) + r_1(x)$ $\quad r_1(x) = f(x^{n/2} = -1)$

$r_1(x) = (a_0 - a_{n/2}) + (a_1 - a_{n/2+1})x + \cdots + (a_{\frac{n}{2}-1} - a_{n-1}) x^{n/2-1}$

We can compute $r_0(x)$ in $n/2$ additions and $r_1(x)$ in $\frac{n}{2}$ subtractions.

Observe that for $\underline{0 \le i < \frac{n}{2}}$

$f(w^{2i}) \overset{r_0}{=} q_0(w^{2i})(\underbrace{w^{2i\cdot\frac{n}{2}}}_{=1} - 1) + r_0(w^{2i})$ Two FFTs of size $n/2$

$f(w^{2i+1}) \overset{r_1}{=} q_1(w^{2i+1}) \cdot (\underbrace{w^{2i\frac{n}{2} + \frac{n}{2}}}_{=-1} + 1) + r_1(w^{2i+1}) = r_1^*(w^{2i})$

where $r_1^*(x) = r_1(wx) = \sum_{i=0}^{n/2-1} (a_i - a_{i+\frac{n}{2}})(wx)^i = \sum_{i=0}^{n/2-1} [(a_i - a_{i+\frac{n}{2}}) w^i] \cdot x^i$

Algorithm FFT2 Input $A = [a_0, a_1, \ldots, a_{n-1}] \in F^n$, $n = 2^k$, $w \in F$.

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if n=1 return A.
T := 1.   B := Array(0..n/2-1)   C ← Array(0..n/2-1).
for i = 0,1,2,...,n/2-1 do
    B_i := A_i + A_{i+n/2}      // B contains r0(x).
    C_i := (A_i - A_{i+n/2}) · T   // C contains r1*(x).
    T := w·T   // T = w^{i+1}
od;
B := FFT(B, n/2, w^2)
C := FFT(C, n/2, w^2)
for i = 0,1,...,n/2-1 do
    A[2i] := B[i]
    A[2i+1] := C[i]
od;
return A.
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Let $T(n)$ be the #mults that FFT2 does.
Assume we've precomputed

$W = [1, w, \ldots, w^{n/2-1}, 1, w^2, w^4, \ldots, 1, w^4, w^8, \ldots, 1, 0]$.

$T(n) = \frac{n}{2} + 2T(\frac{n}{2})$, $T(1) = 0$

$\Rightarrow T(n) = \frac{n}{2} \log_2 n$.