On the Density of Integers Bi-representable as the Sum of Two Cubes

Michael Coons and Paul Vrbik

Department of Mathematics, Simon Fraser University

Abstract

Denote by $\nu(x)$ the number of positive integers less than or equal to x that are representable in at least two ways as the sum of two positive cubes. We address the following question: find the least $\vartheta > 0$ so that

$$\nu(x) \ll x^{\vartheta+\varepsilon}.$$

The best known result is $\vartheta = \frac{4}{9}$. We provide computational evidence for the conjectured asymptotic bound of $\vartheta = \frac{2}{5}$.

Introduction

Consider the integers that are expressible as the sum of two cubes in more than one way. Similarly, one may consider integer solutions to the diophantine equation

$$w^3 + x^3 = y^3 + z^3. \tag{1}$$

Inevitably all recountings of this story involve Hardy, Ramanujan, and the number 1729. And so we wish to get this out of the way directly. Hardy writes in [2],

> I remember going to see [Ramanujan] once when he was lying ill in Putney. I had ridden in taxi-cab No. 1729, and had remarked that the number seemed to me a rather dull one, and that I hoped that it was not an unfavorable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."

Ramnujan's one solution gives infinitely many (though trivial) solutions to Eq. (1). There are many natural questions related to this equation, but our main concern is the question of the asymptotic number of integers that can be written as the sum of two cubes in more than one way.

Definition Denote by $\nu(x)$ the number of positive integers less than or equal to x that are representable in at least two ways as the sum of two positive cubes.

Problem 1 Find the least $\vartheta > 0$ so that

$$\nu(x) \ll x^{\vartheta+\varepsilon}. \tag{2}$$

The symbol "$\ll$" is called the Vinogradov symbol, and is used instead of "Big O" notation.

There are some notable attempts at Problem 1, but before exploring the known results, let us describe probabilistically what is expected.

Probably

For the purpose of finding a probable asymptotic value, let us assume that the events of n being a sum of two cubes once, as well as, twice are independent (though the value is most certainly not independent). Relying on this independence, let us also assume a Poisson distribution, and approximate $\nu(x)$.

Note that the values n such that $n = x^3 + y^3$, with $x^3 + y^3 \le N$ define a region in the plane. Using information from this region, we gain the following expected asymptotic value for $\nu(x)$.

Assuming a Poisson distribution we have that

$$\nu(x) \ll x^{1/3}.$$

That is, assuming a Poisson distribution, $\vartheta = \frac{1}{3}$. But as we stated at the beginning of this section, assuming independence is not very reasonable, and so $\vartheta = \frac{1}{3}$ is most likely not true (as we see in the next section), though it gives us a rough estimate for a starting point.

Known Upper Bounds

Year	Author	ϑ
1963	Hooley	$\vartheta = \frac{2}{3}$
1980	Hooley	$\vartheta = \frac{5}{9}$
1997	Heath-Brown	$\vartheta = \frac{4}{9}$

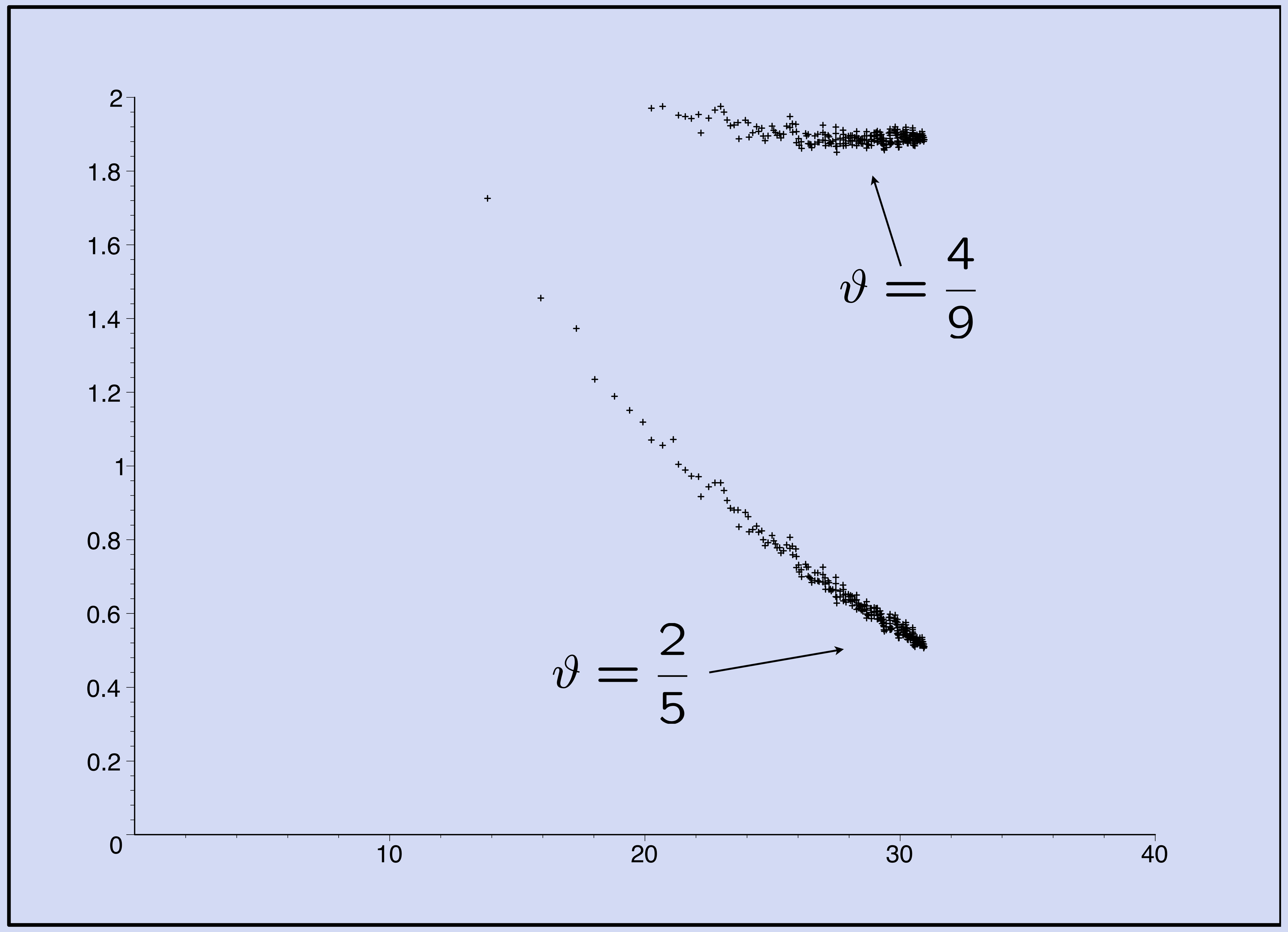

Figure 3: The $\frac{2}{5}$−comet and $\frac{4}{9}$−comet.

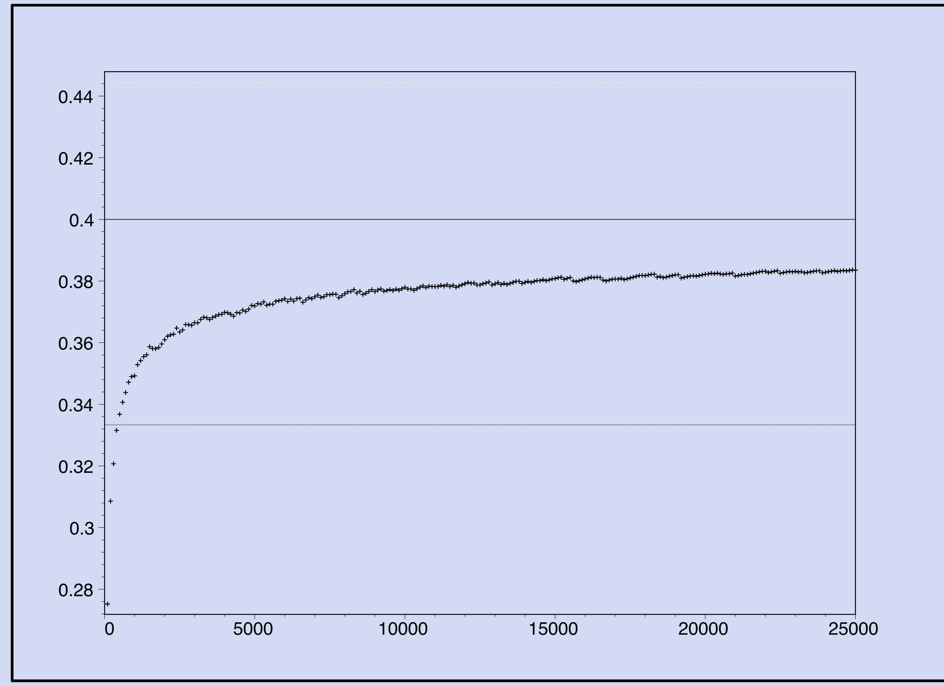

Figure 1: Plot of n versus $\frac{\log \max |K_n|}{\log |K_n|}$.

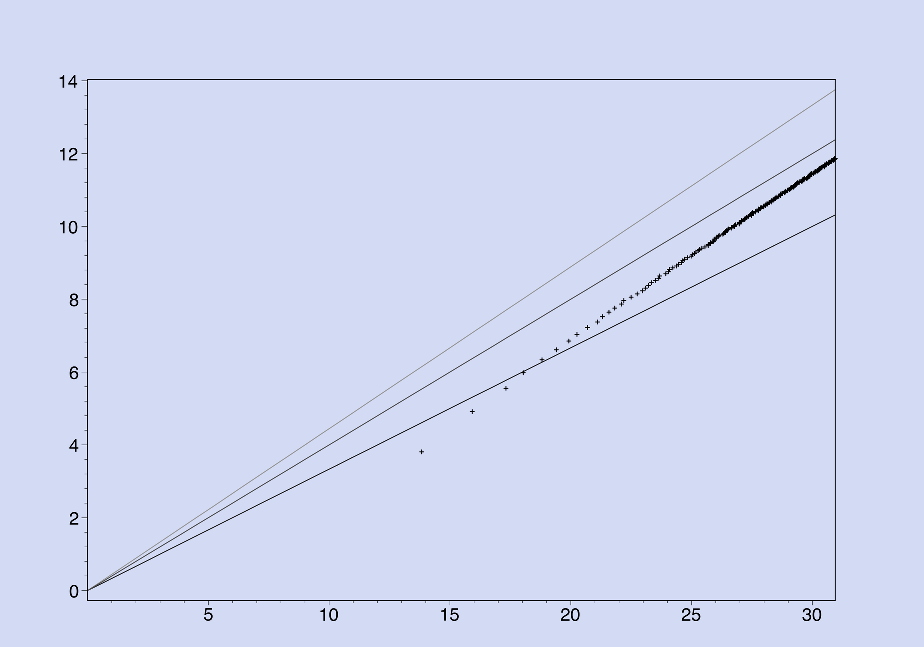

Figure 2: Plot of $\log |K_n|$ versus $\log \max |K_n|$.

Two Conjectures

A very strong conjecture appears in [3], but seems to have originated with Franke, Manin, and Tschinkel [1].

Conjecture 1. For any $\varepsilon > 0$, we have $\nu(x) \ll x^{1/3+\varepsilon}$.

The second conjecture seems to have originated with Stefan Burr of CUNY sometime around October of 2002.

Conjecture 2. For any $\varepsilon > 0$, we have $\nu(x) \ll x^{2/5+\varepsilon}$.

Computational Evidence

Let K_n be the set of integers which are bi-representable as the sum of two cubes where each summand is bounded by n^3. In set notation

$$K_n = \{k \in \mathbb{N} : \exists x_i, x_1^3 + x_2^3 = x_3^3 + x_4^3 = k, x_1 < x_2, x_1 \ne x_3, 0 < x_i \le n\}.$$

Using this notation we have the equivalence $\nu(x) = |K_x|$.

In order to figure out what may be the actual bound of $\nu(x)$, we computed $|K_n|$, $n = 100i$ $(i = 1, 2, 3, \ldots, 250)$. With this notation $|K_{25000}|$ is the number of integers bi-representable as the sum of two cubes with summands less than or equal to $25000^3 = 1.5625 \cdot 10^{13}$.

Note that the value of ϑ may be found using simple techniques to be

$$\frac{\log \max |K_n|}{\log |K_n|} \ll \vartheta.$$

Figure 1 includes a plot of this relationship.

Included in Figure 2 are the lines $\vartheta = \frac{4}{9}, \frac{2}{5}, \frac{1}{3}$, corresponding to Heath-Brown's known result, Burr's conjecture, and the conjecture of Franke, Manin, and Tschinkel [1]. From this picture, the conjecture of $\vartheta = \frac{2}{5}$ seems to be very convincing.

Figure 2 shows $\log |K_n|$ versus $\log \max |K_n|$. The middle line corresponds to $\vartheta = \frac{2}{5}$ and the top line to $\vartheta = \frac{4}{9}$. In this plot the data seems to be bending towards the middle line; that is, the line corresponding to $\vartheta = \frac{2}{5}$.

To see if this is indeed the case, we inspect the difference between the data and the line corresponding to $\vartheta = \frac{2}{5}$. In order to see how good a bound the current known bound of $\vartheta = \frac{4}{9}$ is, we also plot the difference between this line and the data. These relationships are reflected in Figure 3.

Since the information of Figure 3 seems to be reflected by two groups that look like comets, we will refer to them as such. We will call the top comet the $\frac{4}{9}$-comet, and the bottom one the $\frac{2}{5}$-comet, as the top comet corresponds to the difference with the line in the plot of Figure 2 corresponding to $\vartheta = \frac{4}{9}$ and the bottom to the line corresponding to $\vartheta = \frac{2}{5}$.

As is the case with comets, these two are going somewhere, in the asymptotic sense. Their destinations tell us whether they reflect good asymptotic bounds or not. The trail of the $\frac{2}{5}$-comet seems to give us a probable destination of the axis or some other constant. What is maybe more interesting is that the $\frac{4}{9}$-comet seems to have come down and at the end of our calculation is starting to go out toward infinity. This tells us that the bound $\vartheta = \frac{4}{9}$ is almost certainly not sharp, and that $\vartheta = \frac{2}{5}$ is maybe the correct asymptotic bound.

References

1. J. Franke, Y.I. Manin and Y. Tschinkel,*Rational point of bounded height on Fano Varieties,* Invent. Math., 95 (1989), 421-435.
2. G.H. Hardy, Ramanujan: twelve lectures on subjects suggested by his life and work, Chelsea Publishing Company, New York, 1959.
3. D.R. Heath-Brown, *The density of rational points on cubic surfaces*, Acta Arith., **1** (1997), 17-30.
4. C. Hooley, *On the representations of a number as the sum of two cubes,* Math. Z., 82 (1963), 259-266.
5. C. Hooley, *On the numbers that are representable as the sum of two cubes,* J. Reine Angew. Math., 314 (1980), 146-173.