Two new Radial Basis Functions

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Abstract

The 3 most common radial basis functions for two dimensional interpolation are:

1. the Gaussian $\exp\left(-\frac{1}{2}\cdot r^2\right)$
2. the Multiquadric $\mathrm{sqrt}(1+r^2)$
3. the Thin Plate Spline $r^2 \cdot \ln(r)$

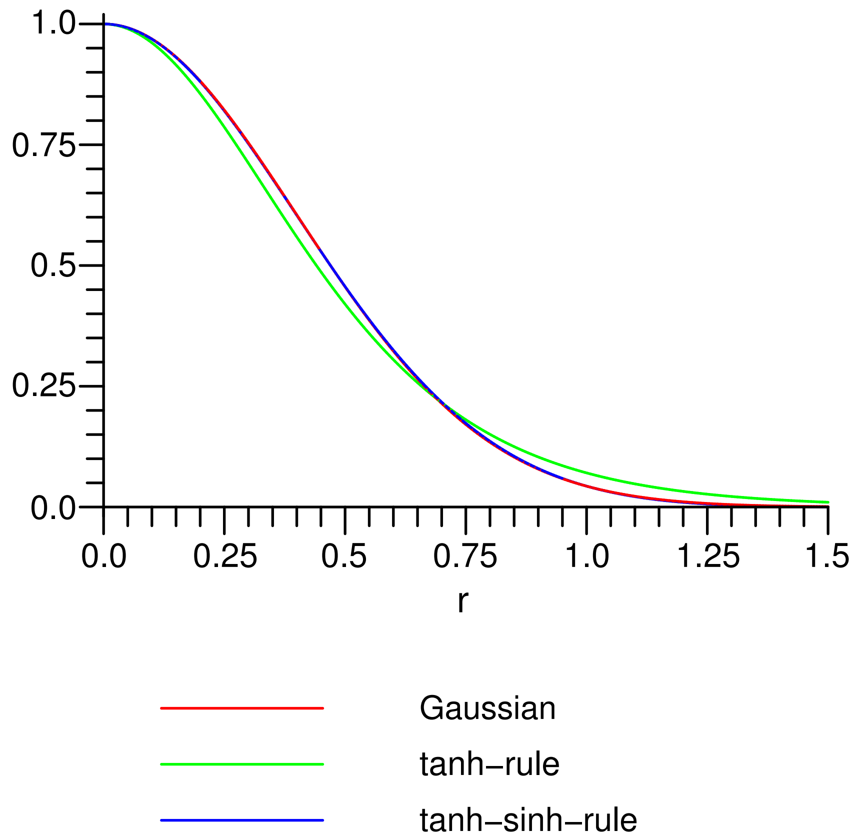
We have discovered 2 others, which are :

4. the tanh–rule weight function $\mathrm{sech}(r)^2$
5. the tanh–sinh–rule weight function $\mathrm{sech}\left(\frac{\pi}{2}\cdot\sinh(r)\right)^2\cdot\cosh(r)$

Introduction

The most common radial basis function is the Gaussian. One can view the Gaussian as the weight function for the erf–rule quadrature formula. The weight function is the derivative of the variable transformation function. We noticed that the weight functions for the tanh–rule and the tanh–sinh–rule quadrature formulas also look like Gaussian curves. We may introduce a scale parameter R by replacing r with $\frac{r}{R}$ in the above formulas. We have chosen scale parameters sc

all 3 functions have the same definite integral. The chosen values are: for the Gaussian $R = (2\cdot\pi)^{\left(\frac{-1}{2}\right)}$, for the tanh–rule $R = \frac{1}{2}$, for the tanh–sinh–rule $R = \frac{\pi}{4}$.



Interpolation Conditions

Given N distinct data points $(x[1], y[1])$, $(x[2], y[2])$, ... $(x[N], y[N])$ in the plane and corresponding heights $z[1], z[2], \ldots z[N]$. Choose a radial basis function $u(r)$. The form of the radial basis interpolatior function is

$$g(x, y) = \sum_{j=1}^{N} c[j] \cdot u\left(\left((x-x[j])^2 + (y-y[j])^2 \right)^{\left(\frac{1}{2}\right)} \right)$$

The N interpolation conditions are:

$$z[i] = \sum_{j=1}^{N} c[j] \cdot u\left(\left((x[i]-x[j])^2 + (y[i]-y[j])^2 \right)^{\left(\frac{1}{2}\right)} \right)$$

for i from 1 to N. We need solve a dense N by N symmetric linear system of equations

Radial basis function procedures

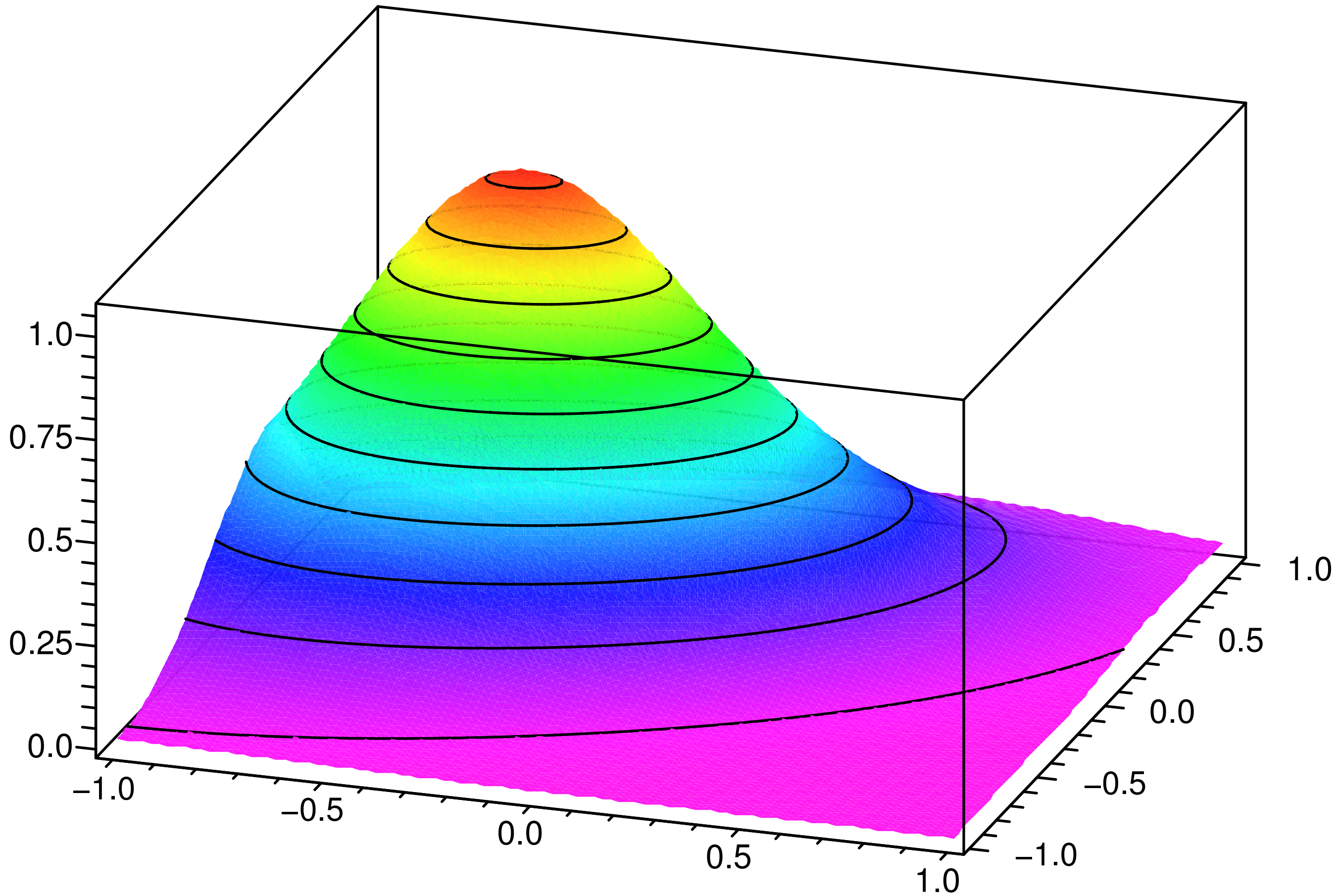
We choose the global variable R as our scale parameter, and define global variables R1:=1/R; and R2:= R1^2;

```
> rbf[1] := proc(r) exp(-1/2*R2*r^2) end proc:
> rbf[2] := proc(r) (1+R2*r^2)^(1/2) end proc:
> rbf[3] := proc(r) local R1r; if r=0 then 0 else R1r := R1*r; R1r^2*ln(R1r) end
  if end proc:
> rbf[4] := proc(r) sech(R1*r)^2 end proc:
> rbf[5] := proc(r) local R1r; R1r := R1*r;cosh(R1r)*sech(evalf(Pi)/2*sinh(R1r))^2
  end proc:
```

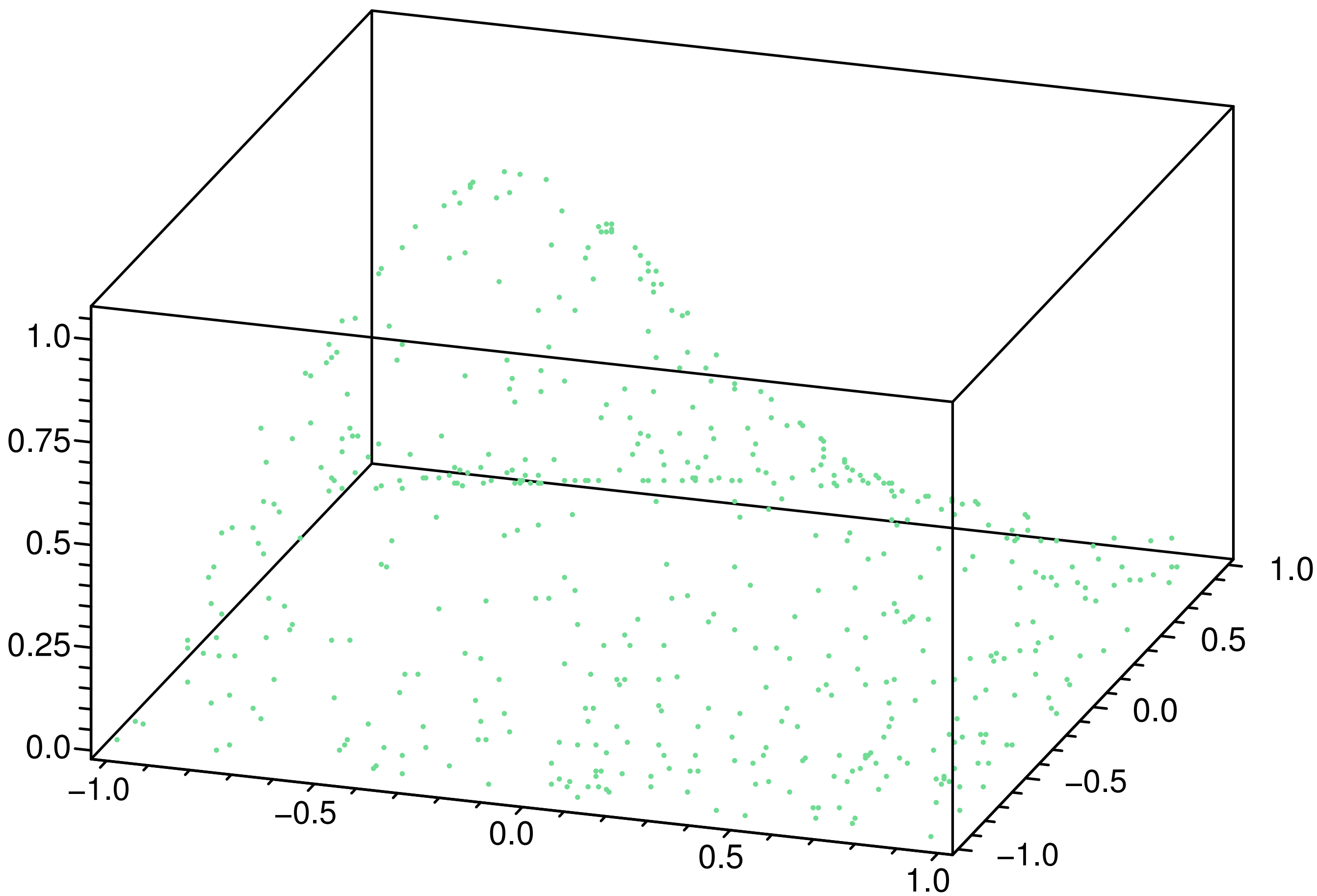
Exact function

```
> exactf := proc(x,y) local x1,y1; x1:=x+1/4; y1:=y-1/6; exp(-2*x1^2+3*x1*y1-3*
  y1^2)*(1+x1+x1^2+2*x1*y1+5*y1^2)^(-3/4); end proc:
```

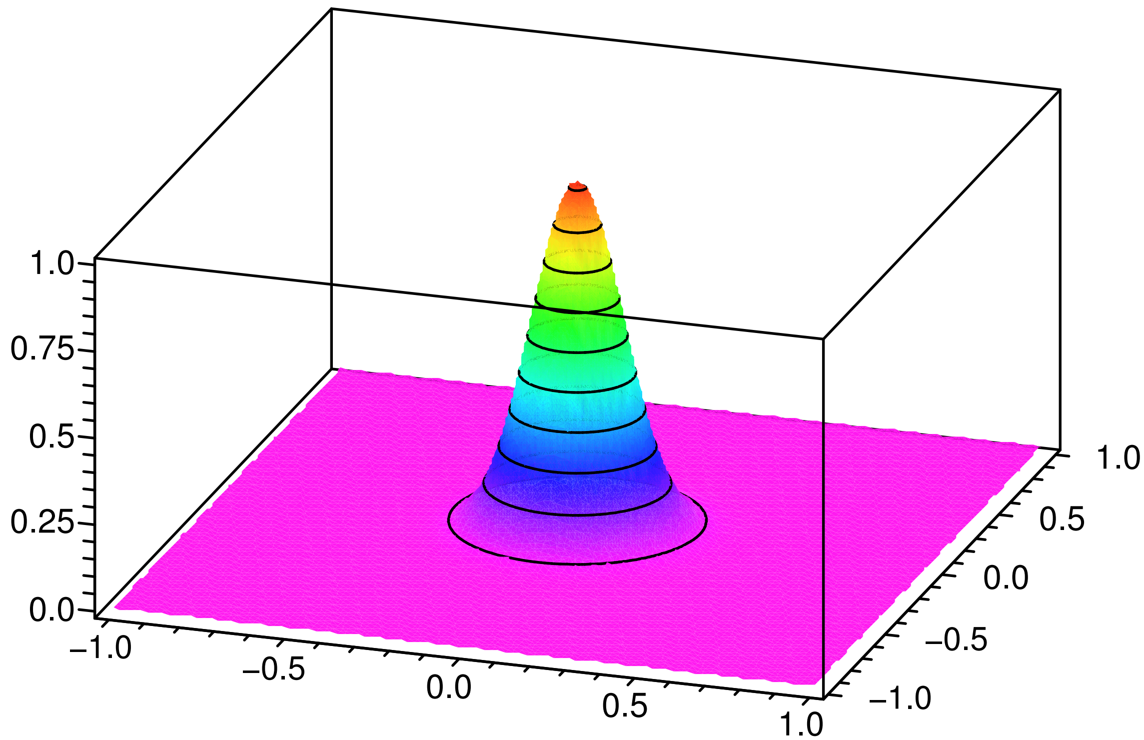
exact function



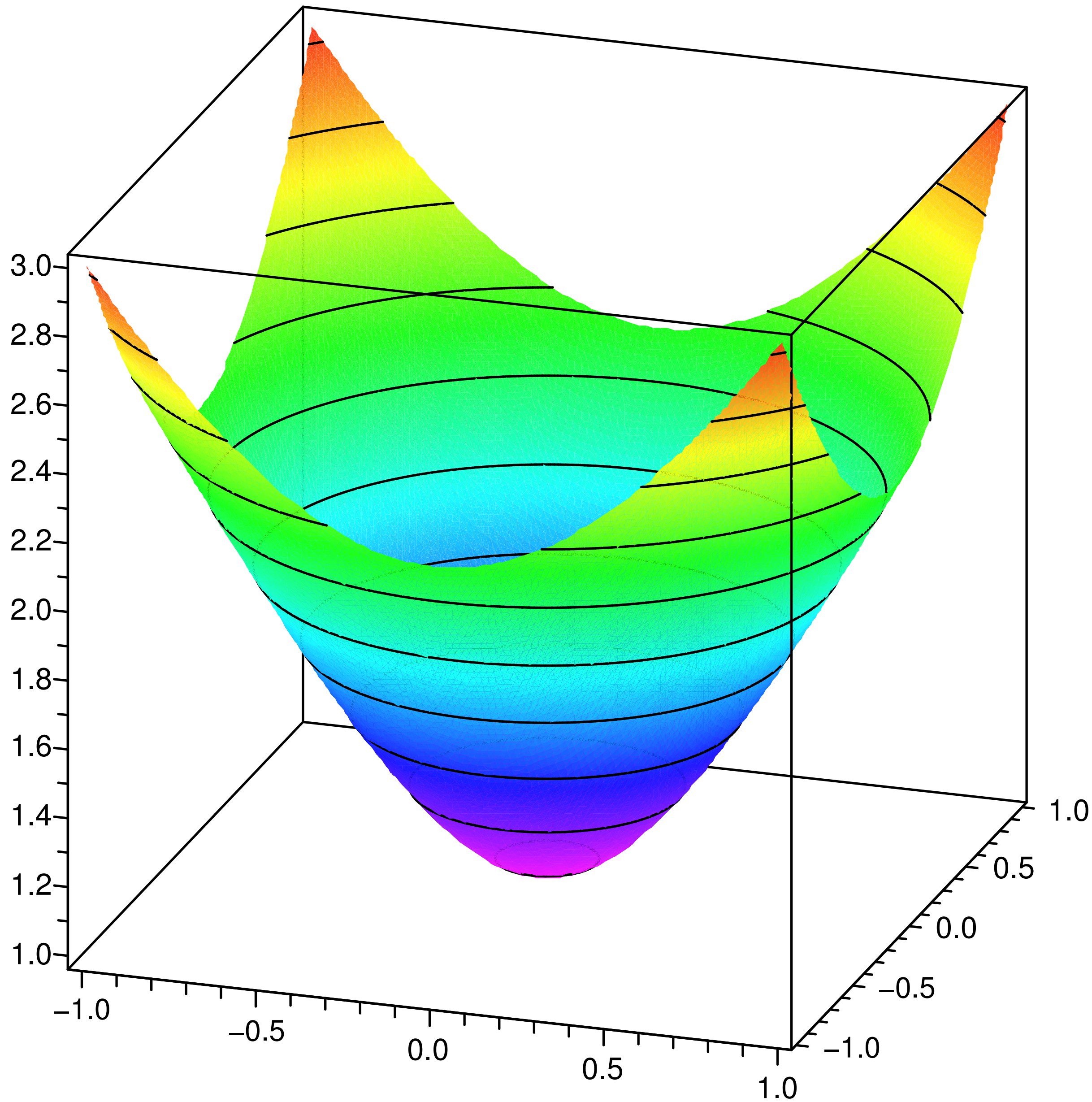
512 uniformly random data points



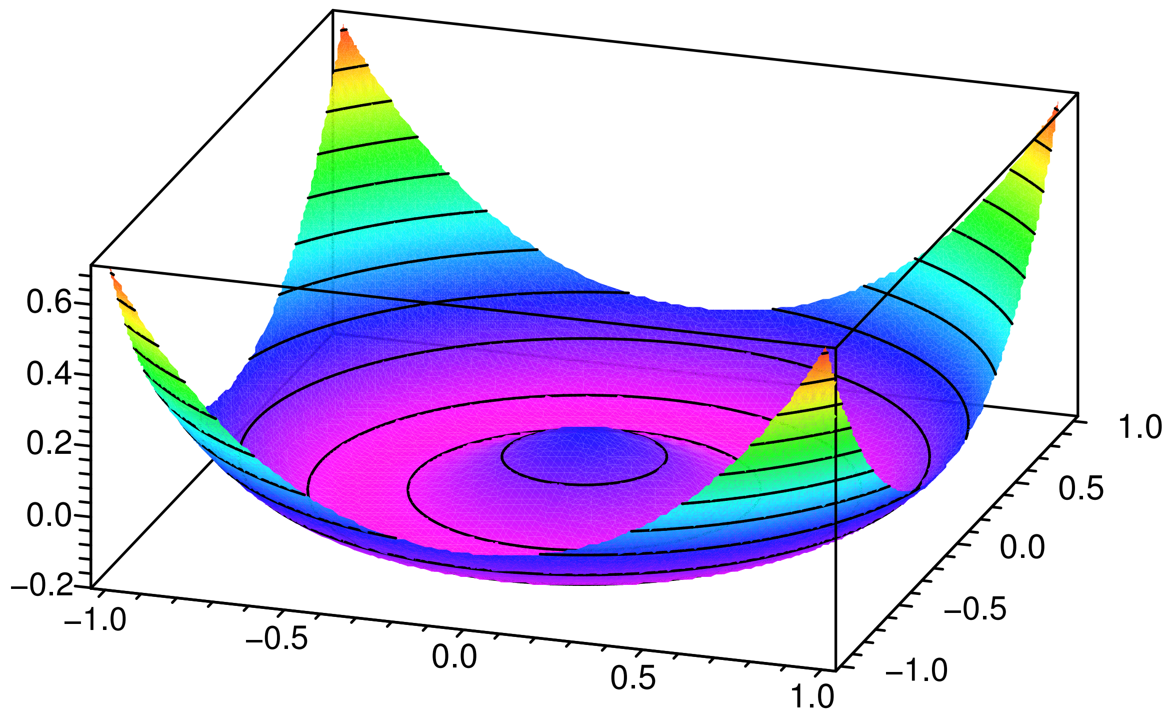
Gaussian radial basis function, R=0.125



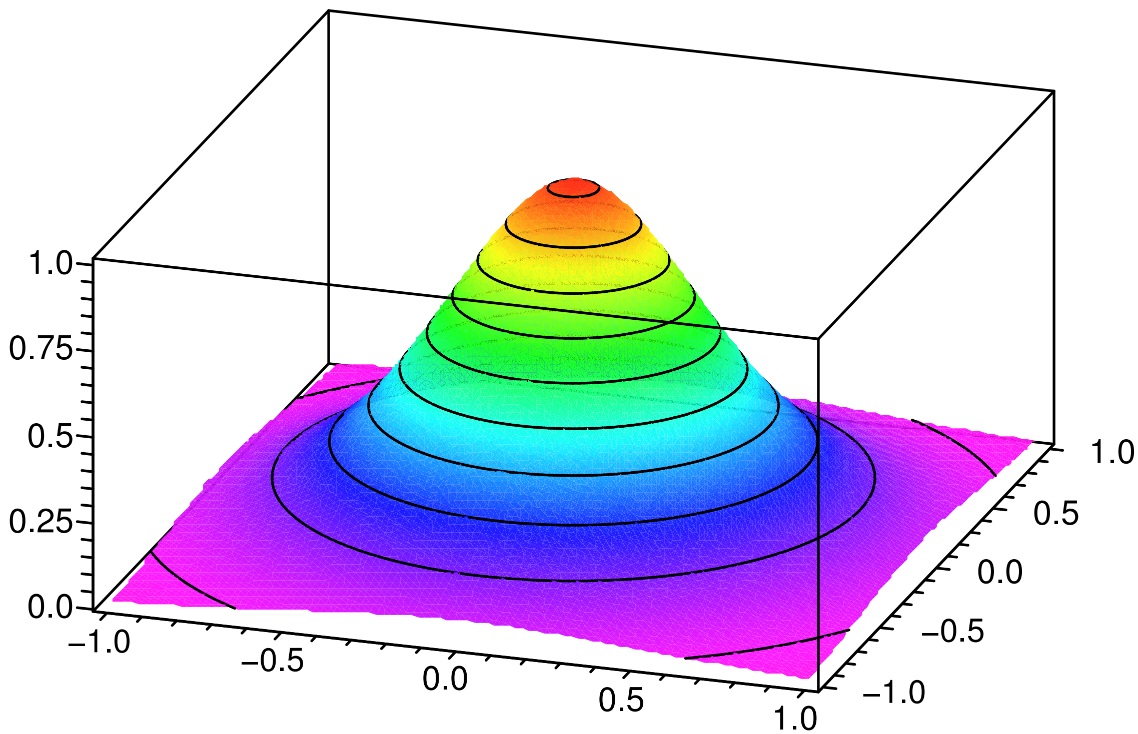
multiquadric radial basis function, R=0.5



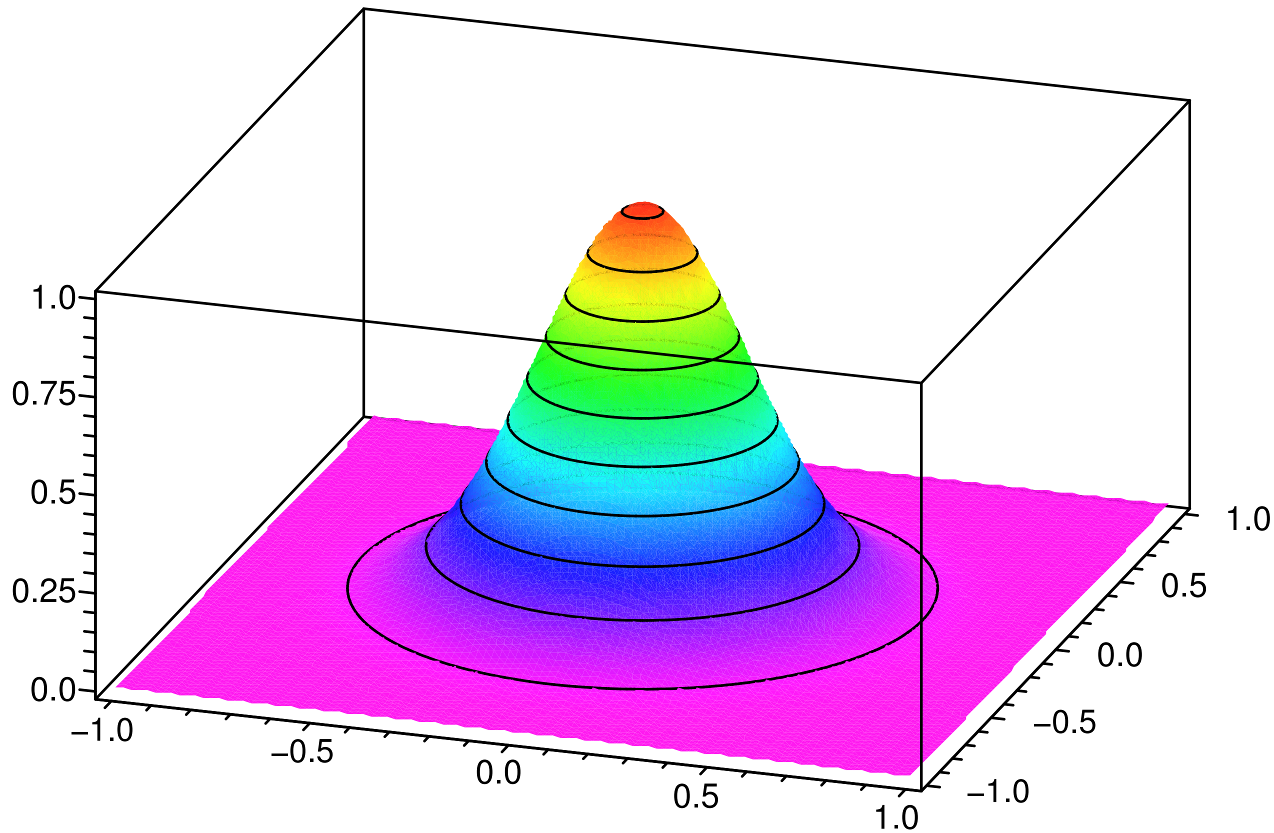
thin plate spline radial basis function, R=1



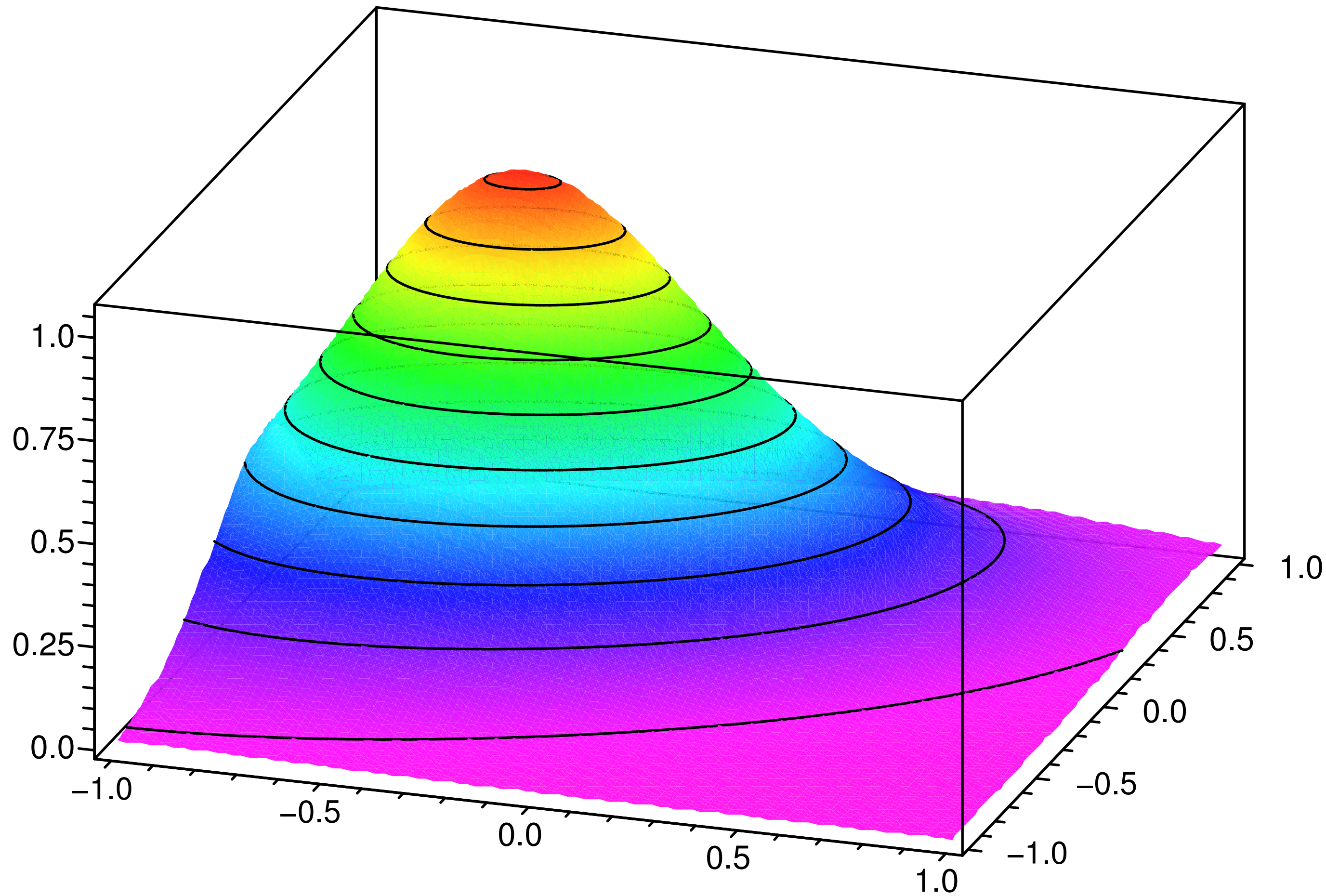
tanh-rule radial basis function, R=0.5

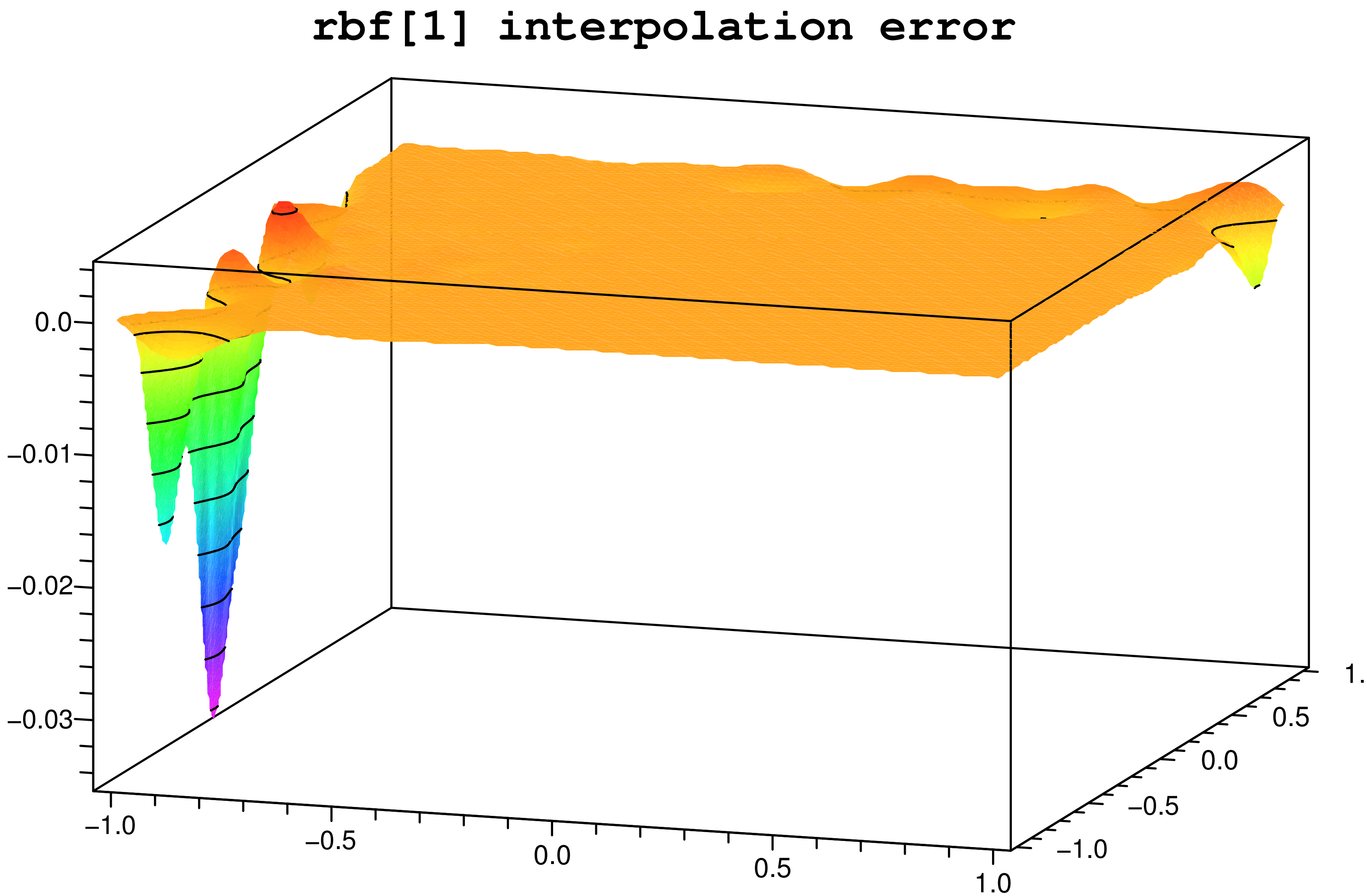


tanh-sinh-rule radial basis function, R=0.5



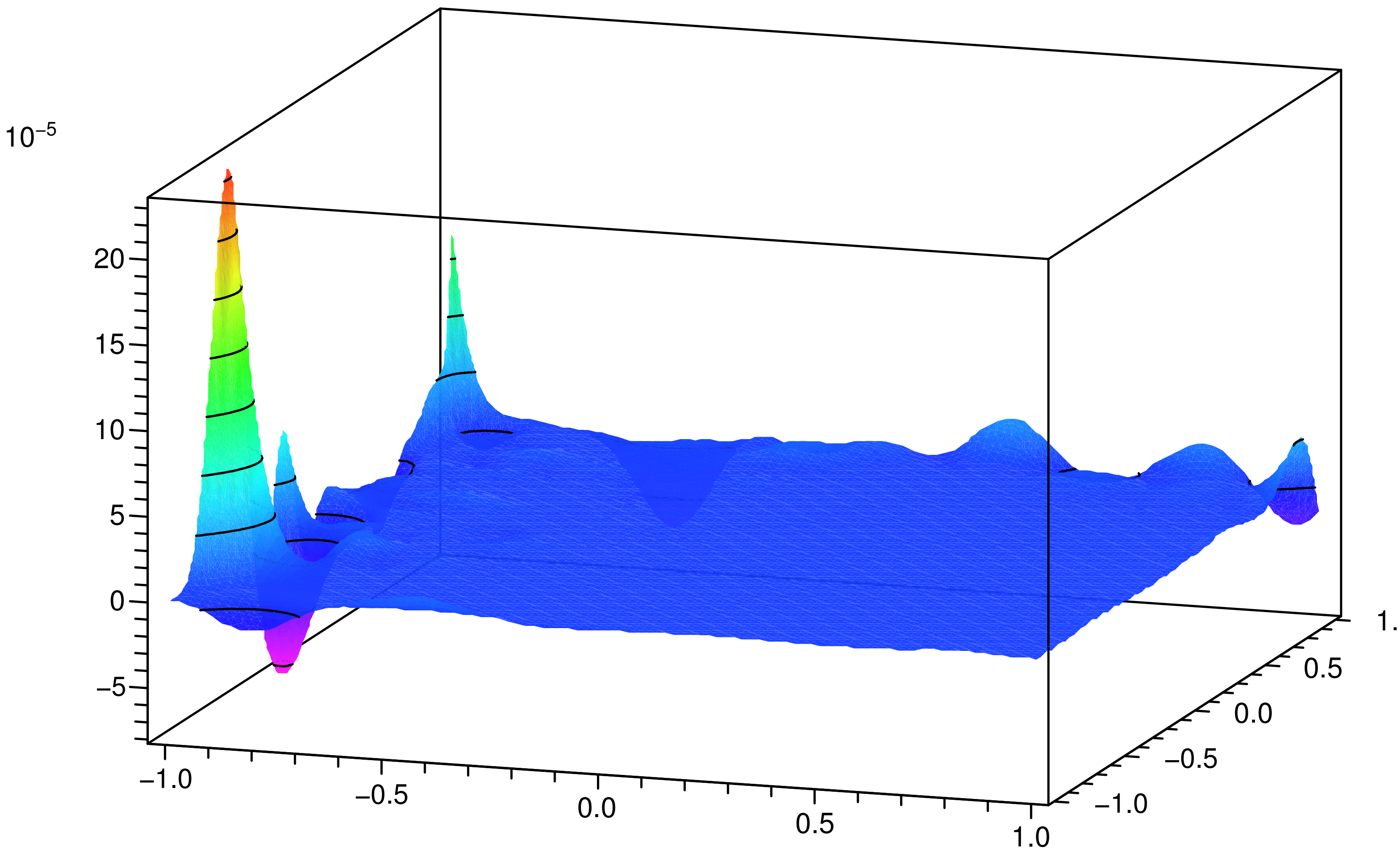
tanh-sinh-rule RBF interpolation



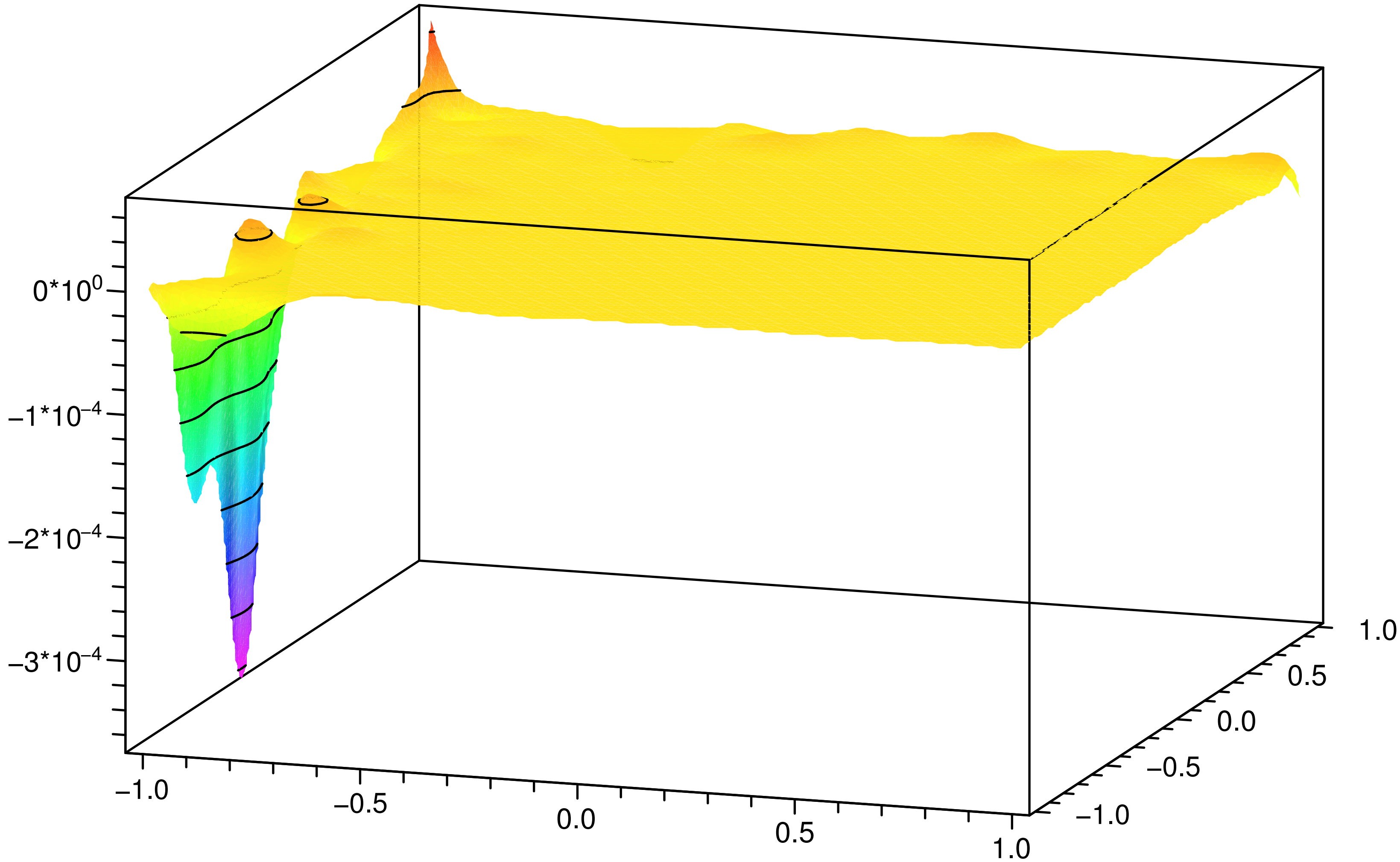


rbf[1] interpolation error
0.0
−0.01
−0.02
−0.03
−1.0
−0.5
0.0
0.5
1.0
1.
0.5
0.0
−0.5
−1.0

rbf[4] interpolation error



rbf[5] interpolation error



Conclusions

Here is a table of the 2–norm of the interpolation error for each the 5 choices of radial basis function.

RBF	*||error||*
1	0.00184363
2	0.00001448
3	0.00218664
4	0.00001266
5	0.00001088

We can see from the above table that our two new radial basis functiions had the least error.

References

David H. Bailey, Karthik Jeyabalan and Xiaoye S. Li, *A Comparison of three high-precision quadrature schemes*, Experimental Mathematics, vol. 14, no. 3, pages 317–329, 2005, **http://crd.lbl.gov/~dhbailey/dhbpapers/index.html** .

Rolland L. Hardy, *Multiquadric Equations of Topography and Other Irregular Surfaces*, Journal of Geophysical Research, Vol. 76, No. 8, pages 1905–1915, March, 1971.

Shmuel Rippa, *An Algorithm for selecting a good value for the parameter c in radial basis function interpolation*, Advances in Computation Mathematics, Vol. 11, Numbers 2–3, pages 193–210, June, 1999.

Bengt Fornberg and Natasha Flyer, *Accuracy of radial basis function interpolation and derivative approximations on 1–D infinite grids*, **amath.colorado.edu/faculty/fornberg/Docs/RBF.pdf**